Delay Insensitive Circuits

an exercise in formal methods

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Plumstead Publishing House

Overview

Delay insensitive circuits are understandable by:

(a) intuitive unconvincing hand-wavy descriptions
(b) formal theories checkable by programs or proofs

Let’s try to find a way from (a) to (b).

The intuitive approach

Executive summary

All you need to know about delay insensitive circuits:

- they have no clocks
- handshake signals and causal relationships make them go
- theory taught in school doesn't work on them
- they are feared, reviled, ignored, and misunderstood

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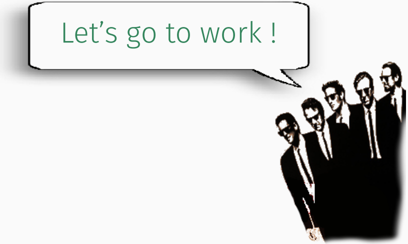
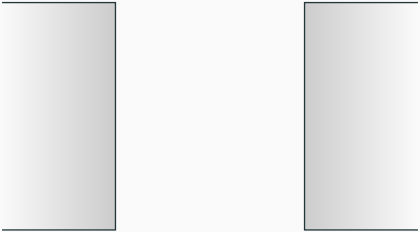


Synchronization by handshakes



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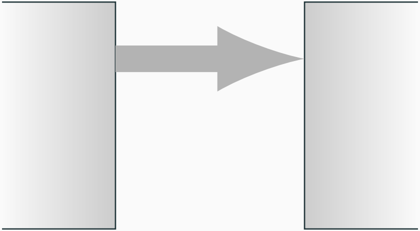
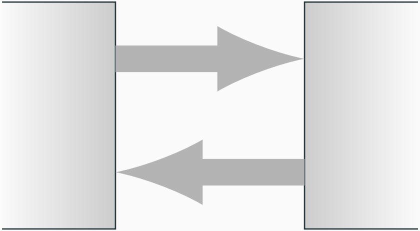


- request

Synchronization by handshakes



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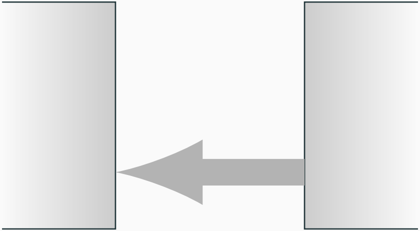

- request
- acknowledge
- release

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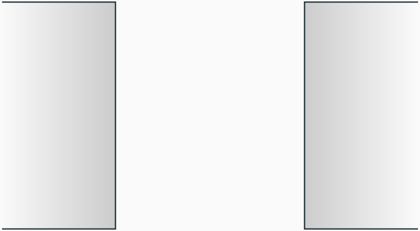

- request
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- unacknowledge

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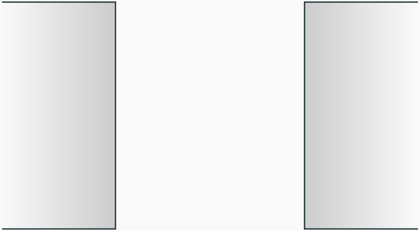

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} unnecessary ? (release, unacknowledge)

Synchronization by handshakes

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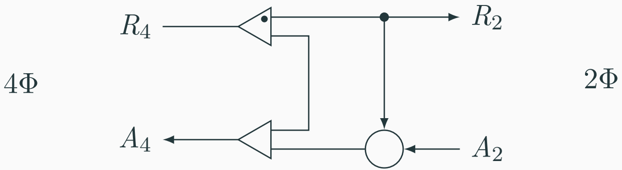

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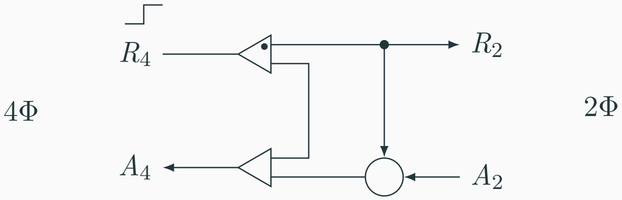

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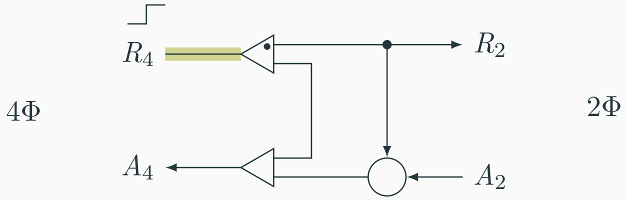

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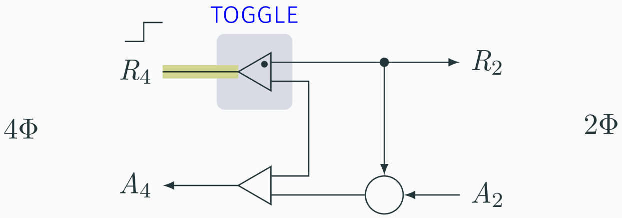

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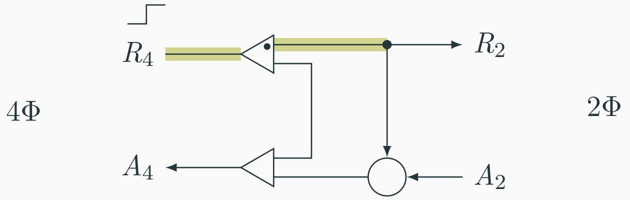

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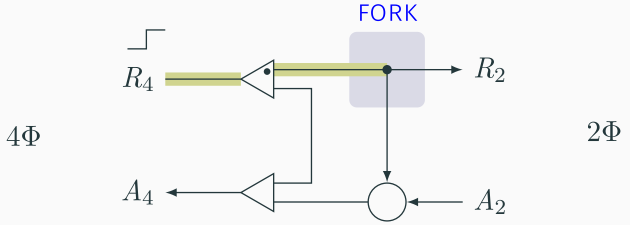

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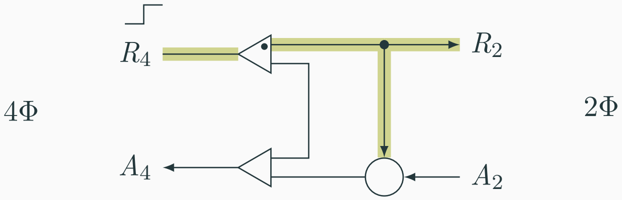

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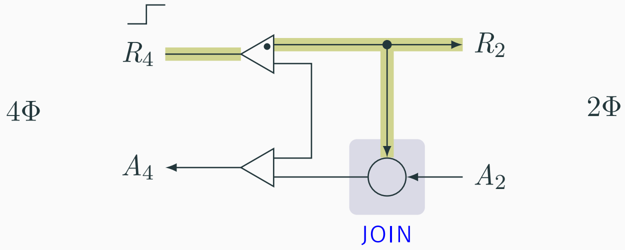

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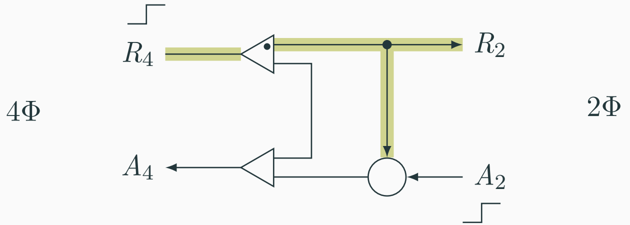

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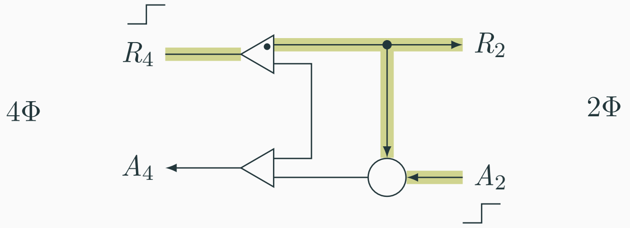

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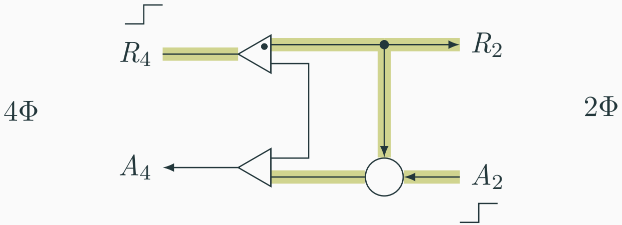

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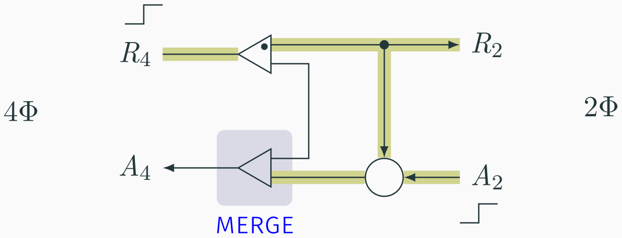

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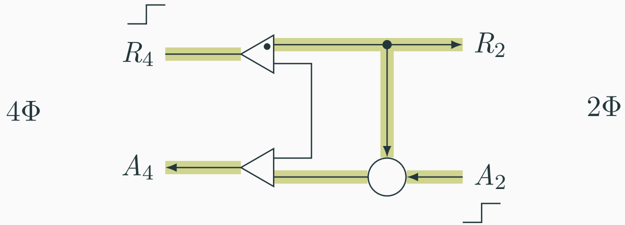

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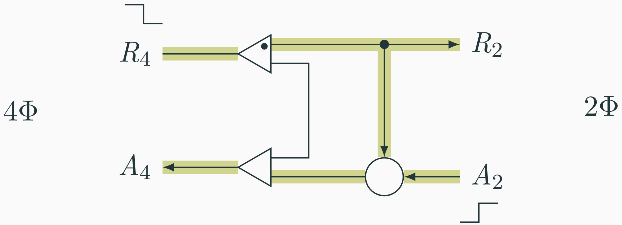

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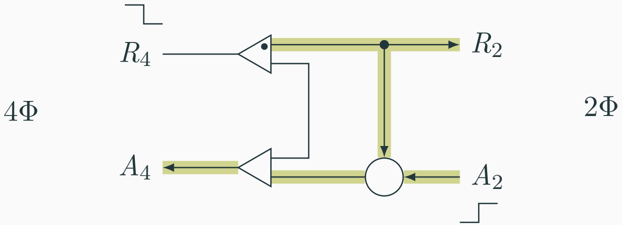

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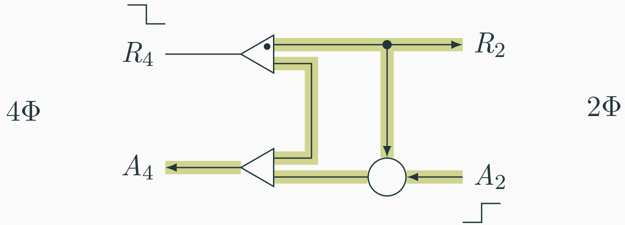

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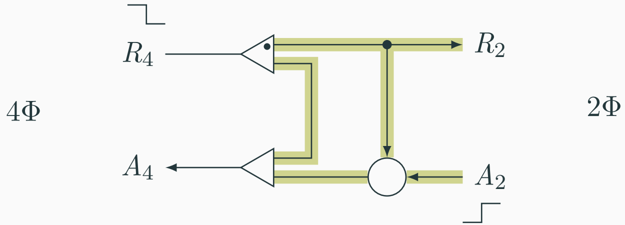

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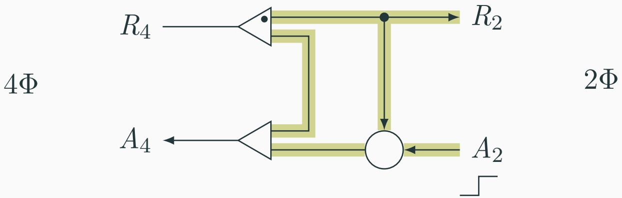

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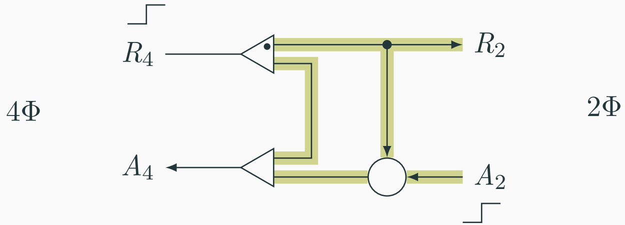

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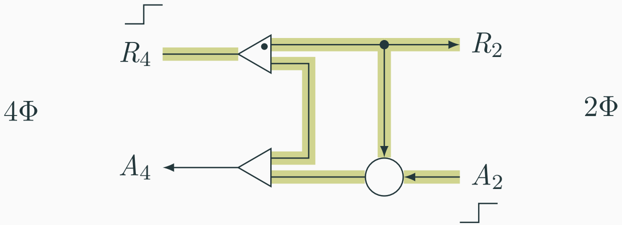

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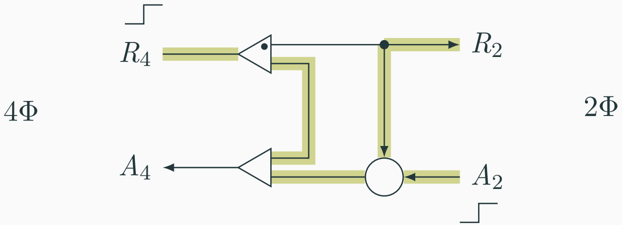

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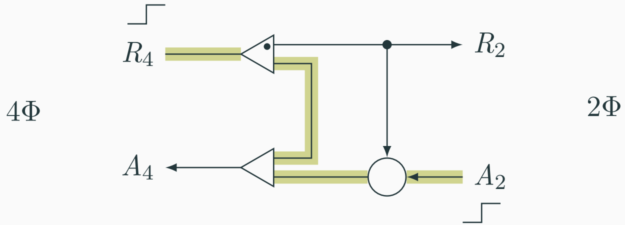

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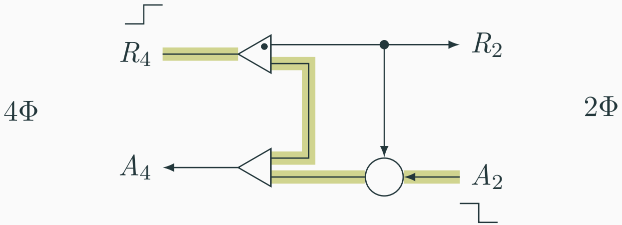

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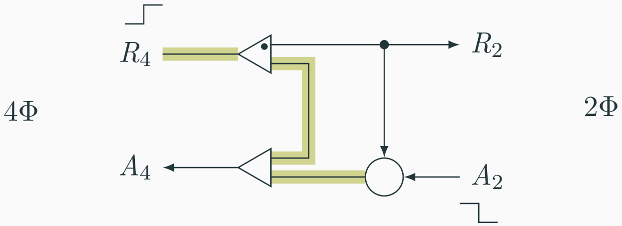

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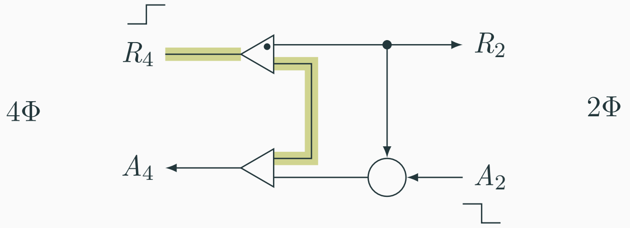

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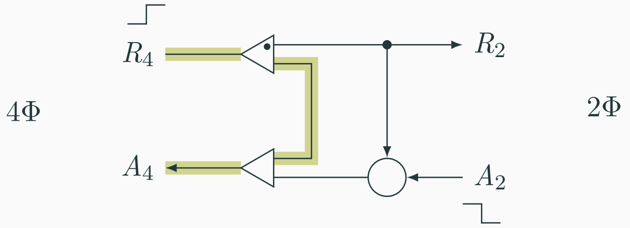

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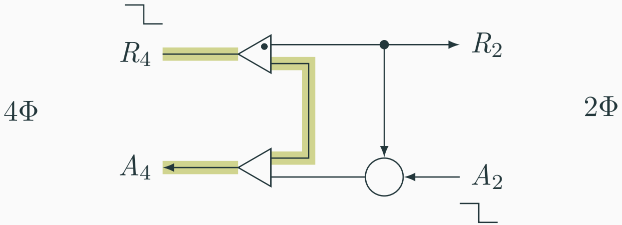

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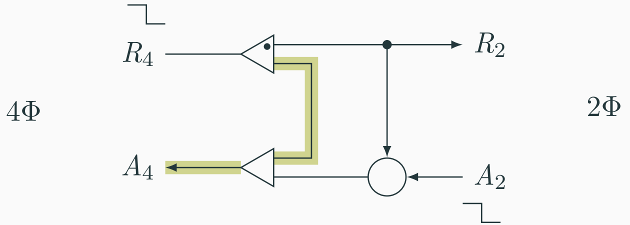

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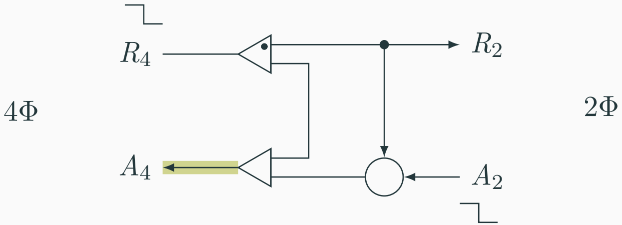

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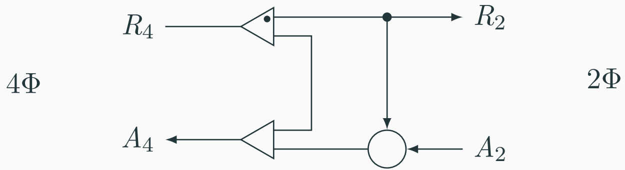

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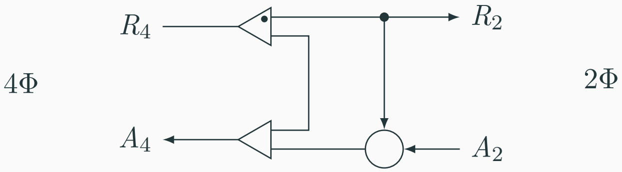

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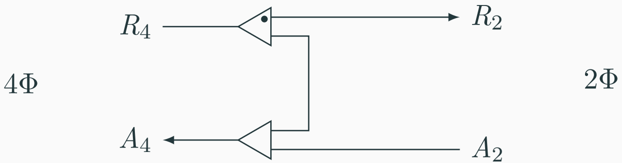

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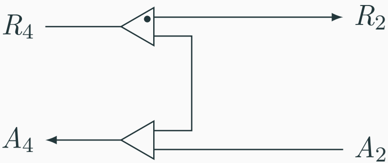

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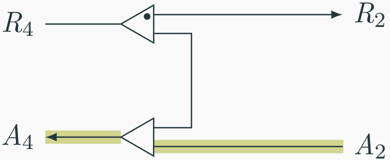

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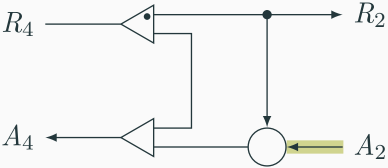

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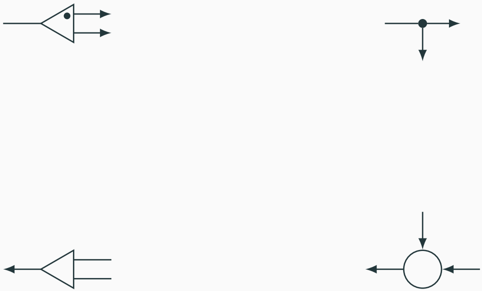

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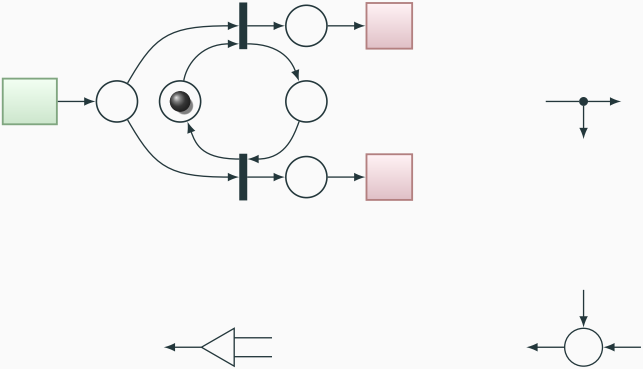

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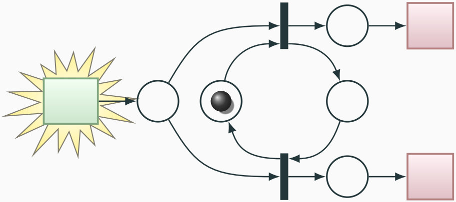

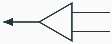

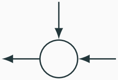

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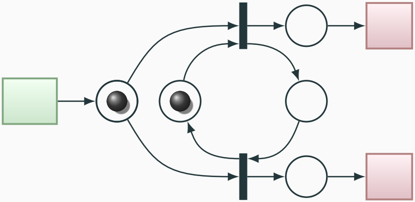

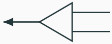

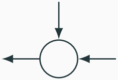

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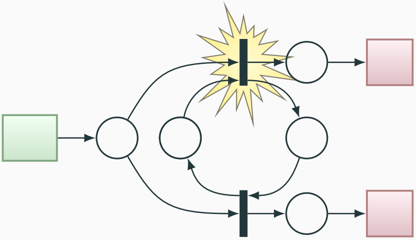

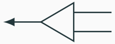

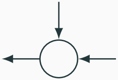

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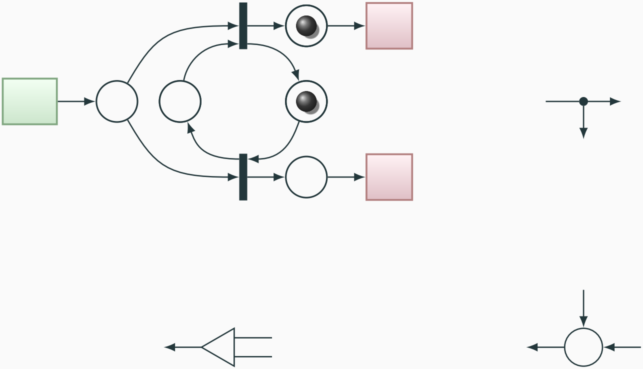

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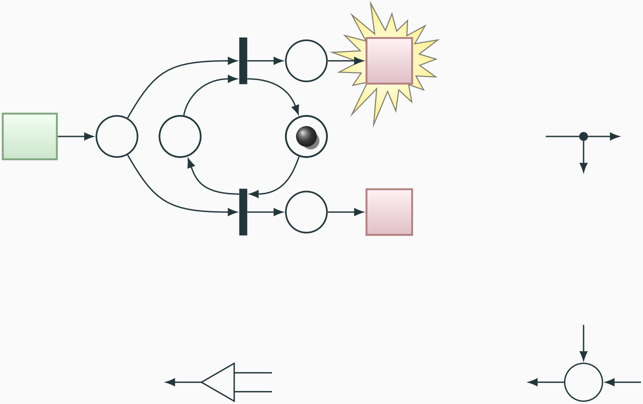

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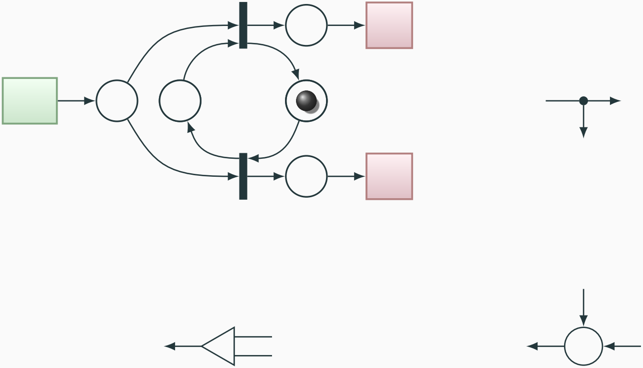

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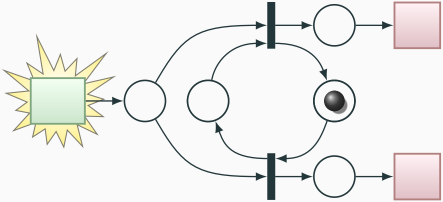

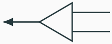

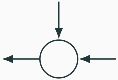

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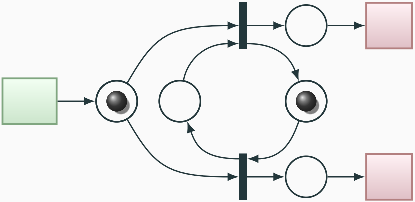

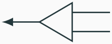

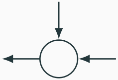

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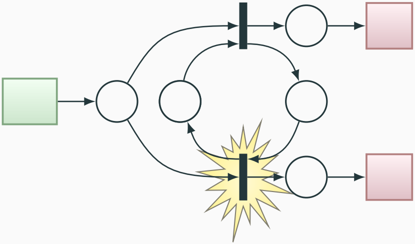

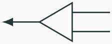

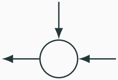

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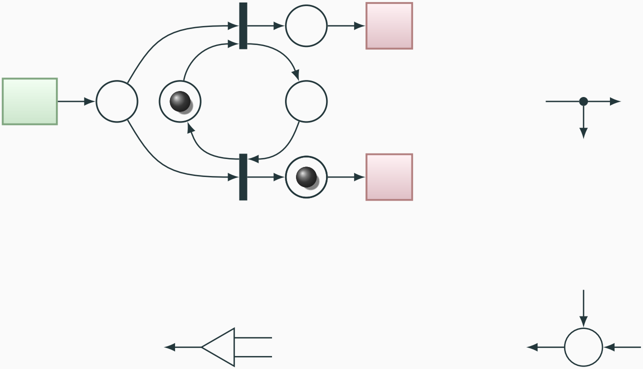

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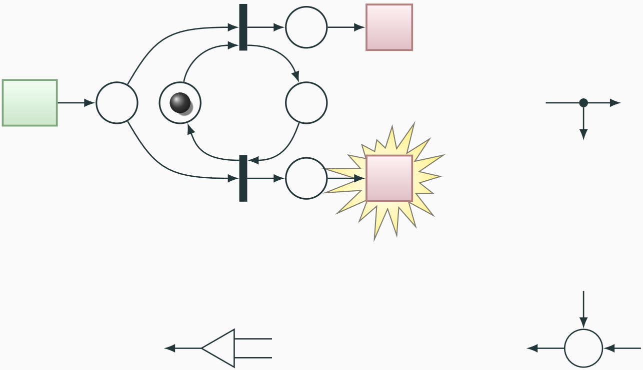

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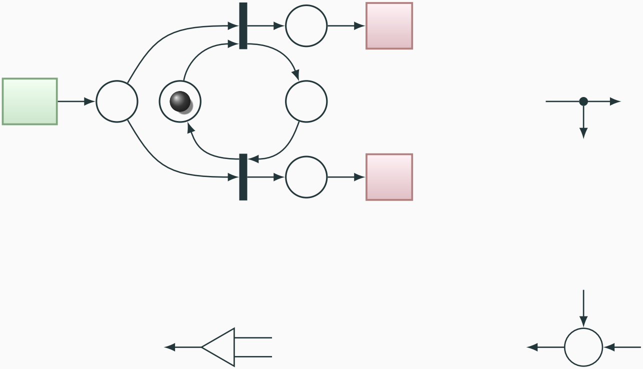

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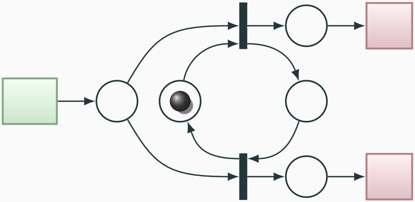

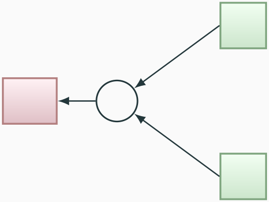

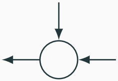

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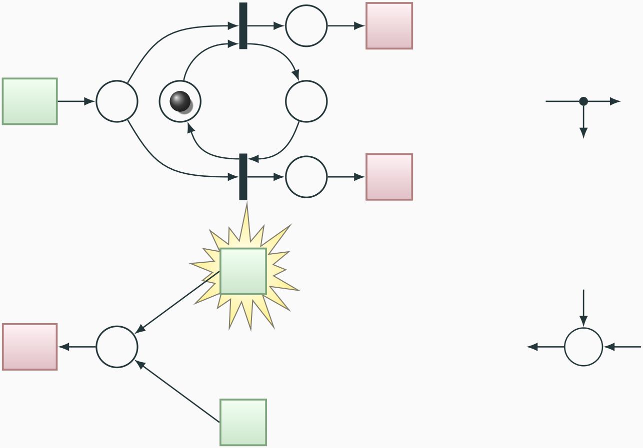

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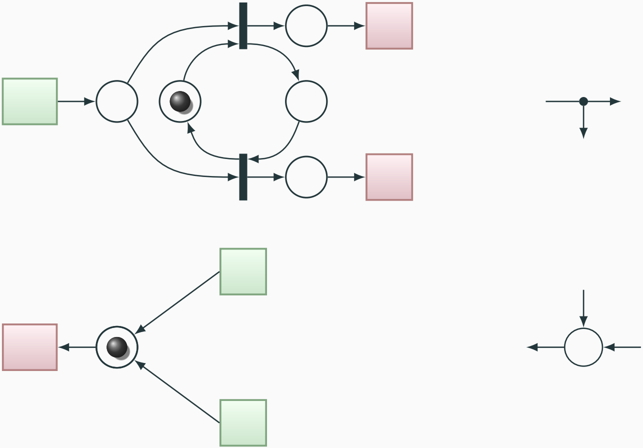

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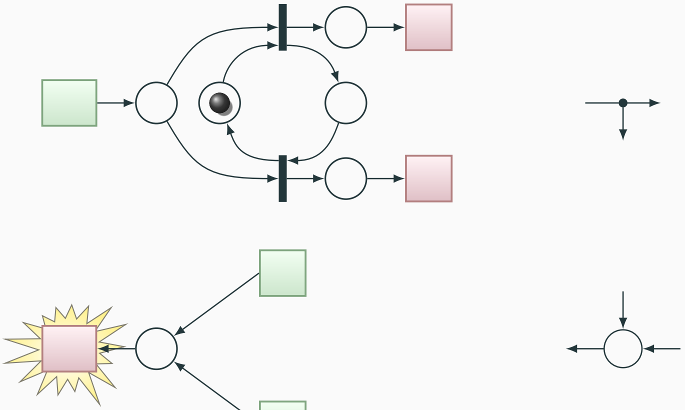

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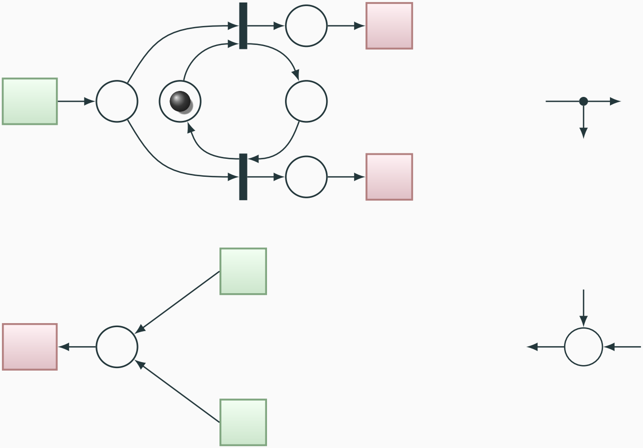

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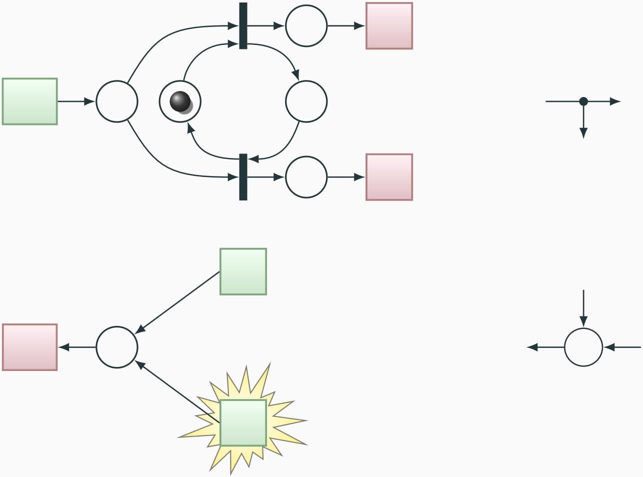

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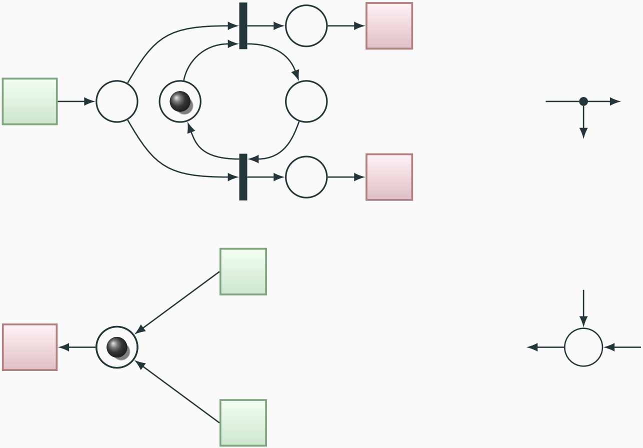

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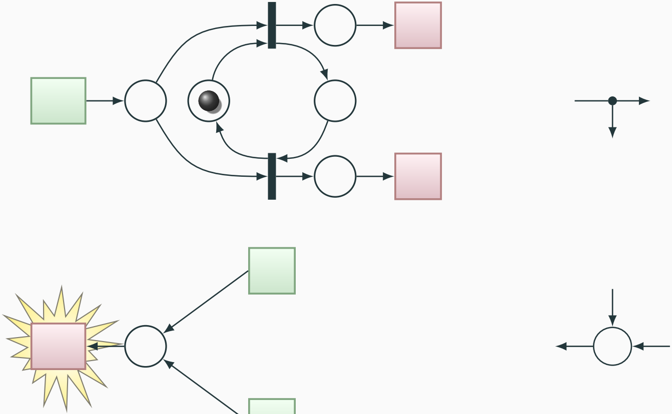

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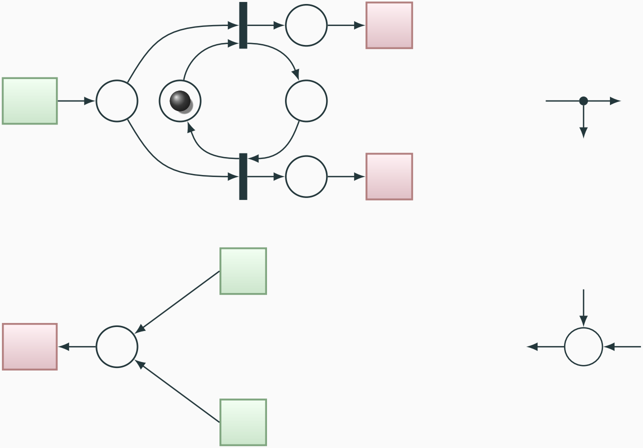

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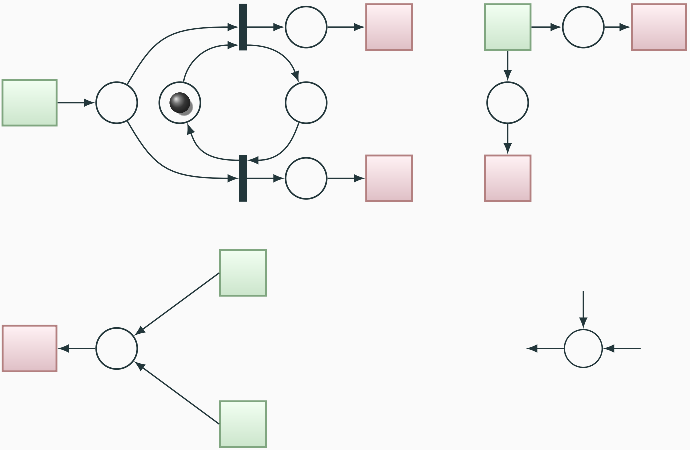

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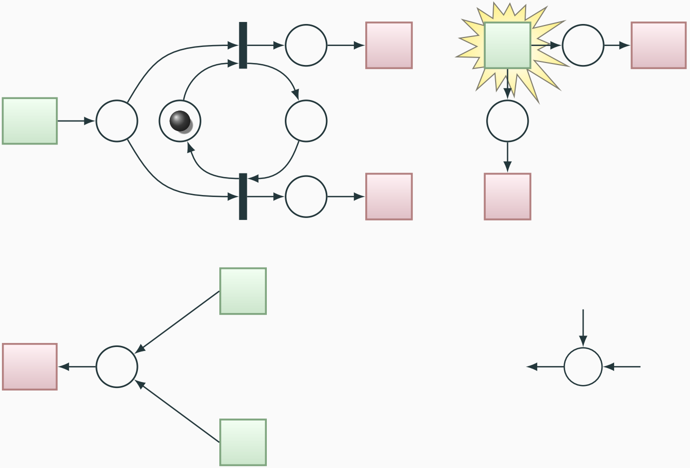

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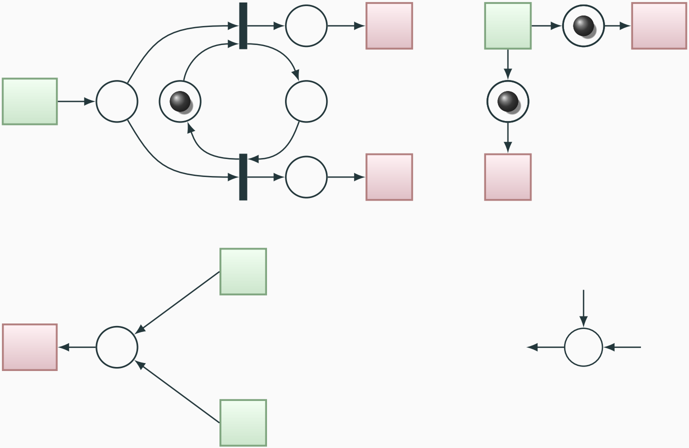

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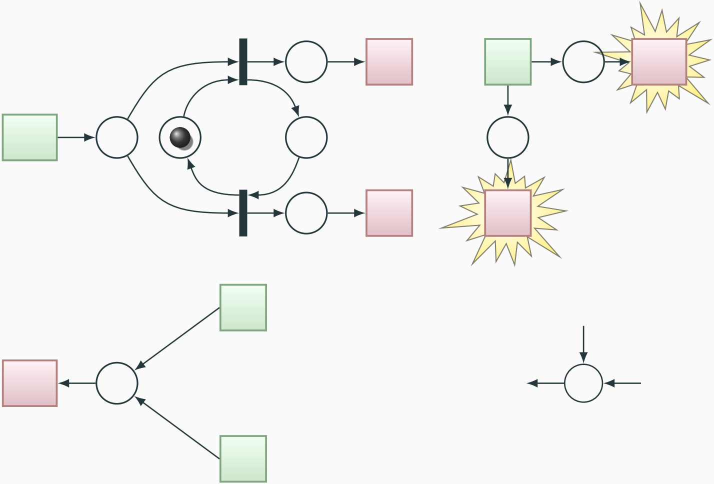

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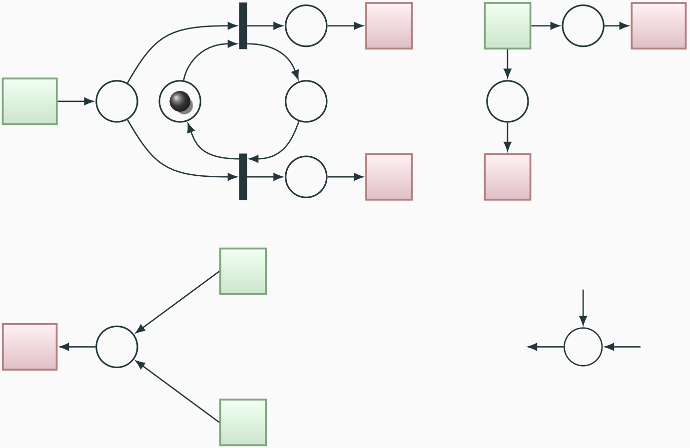

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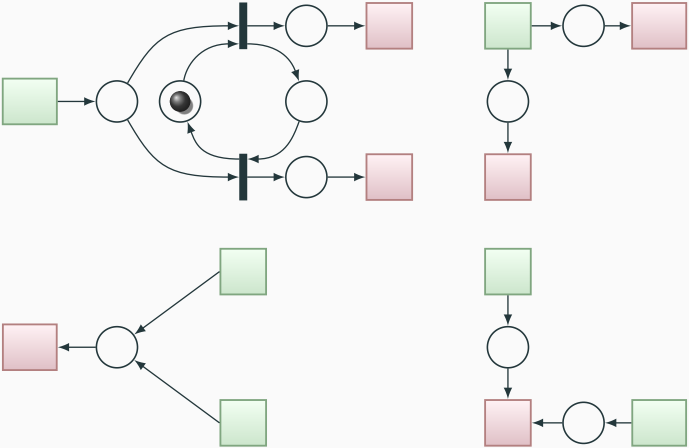

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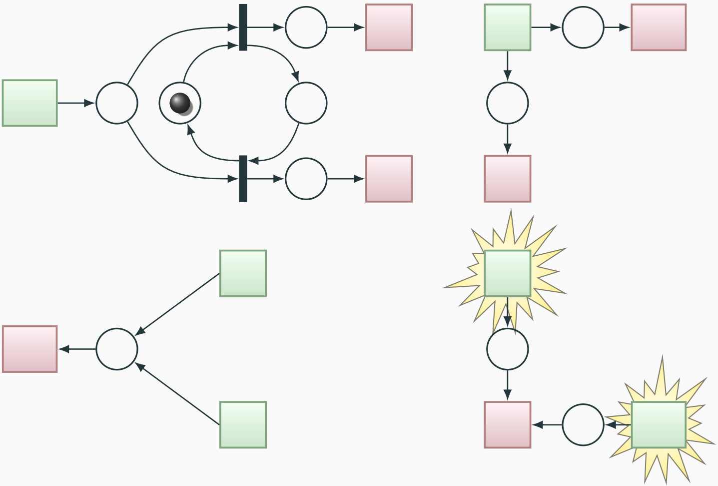

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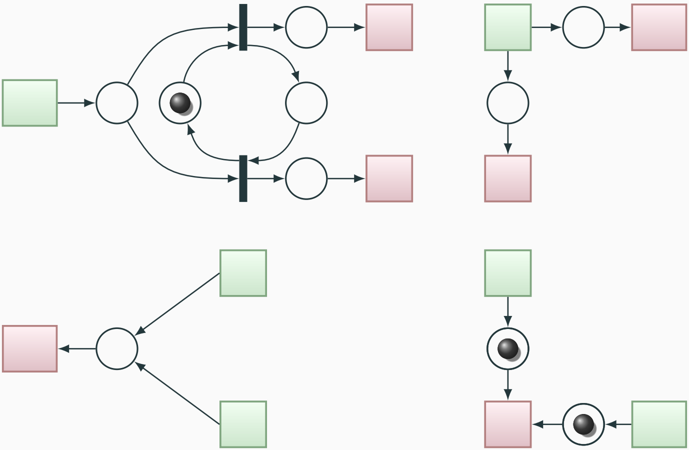

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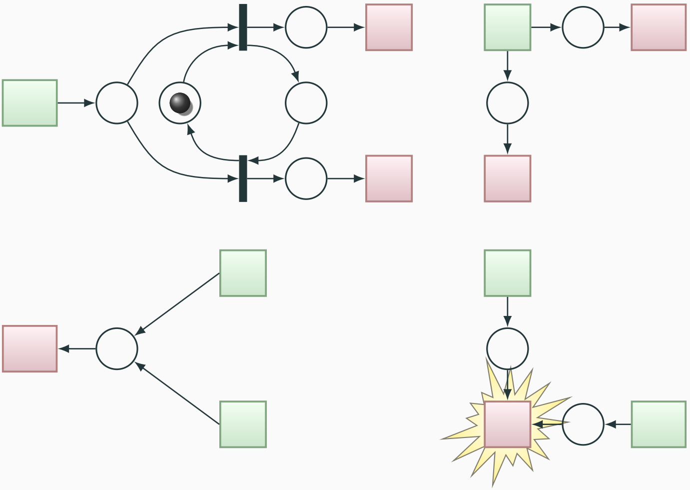

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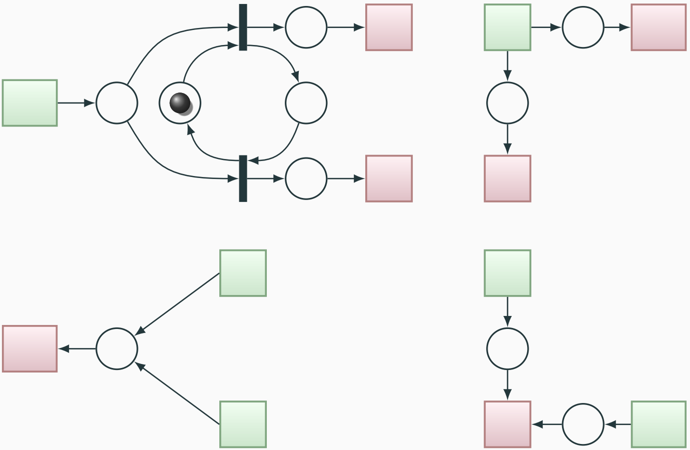

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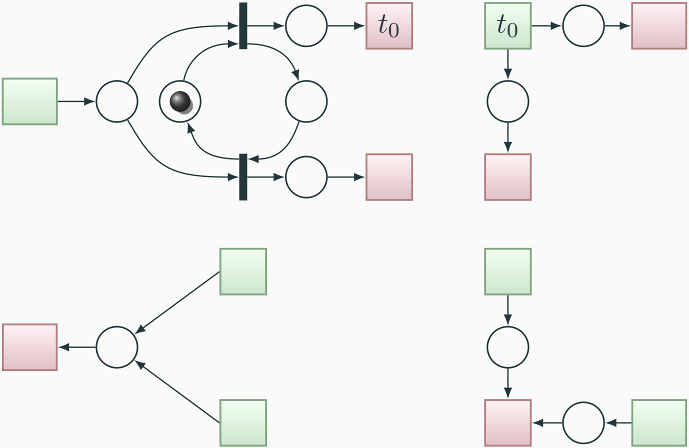

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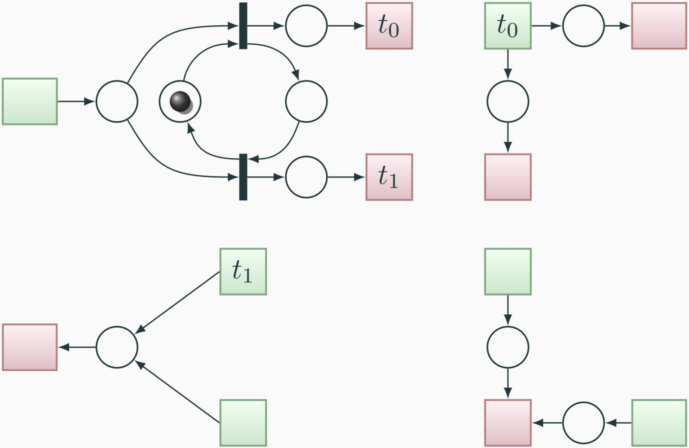

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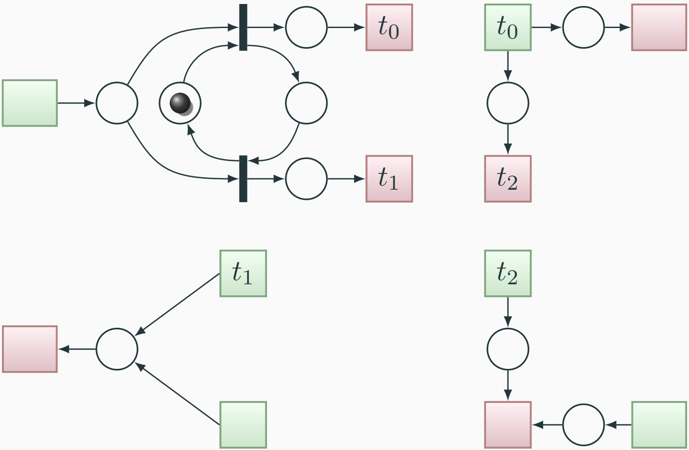

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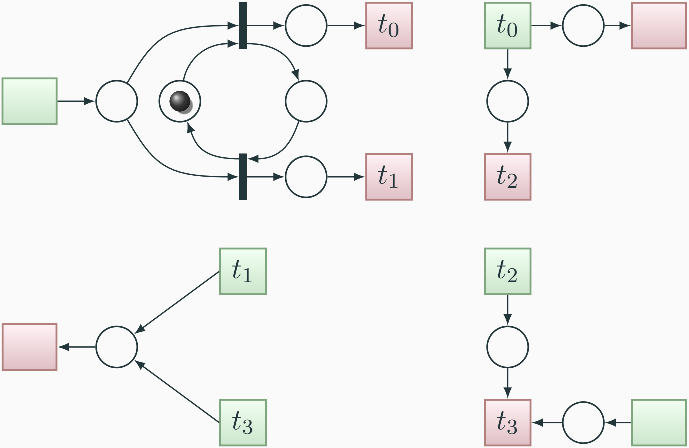

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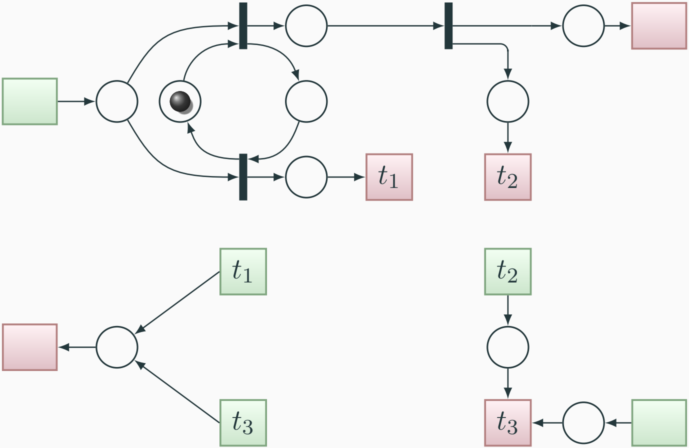

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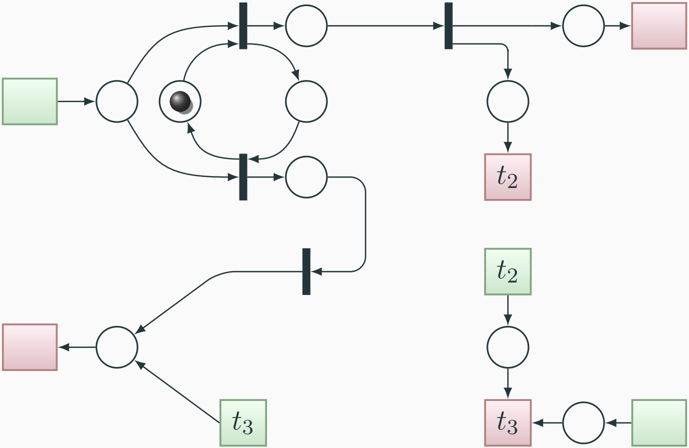

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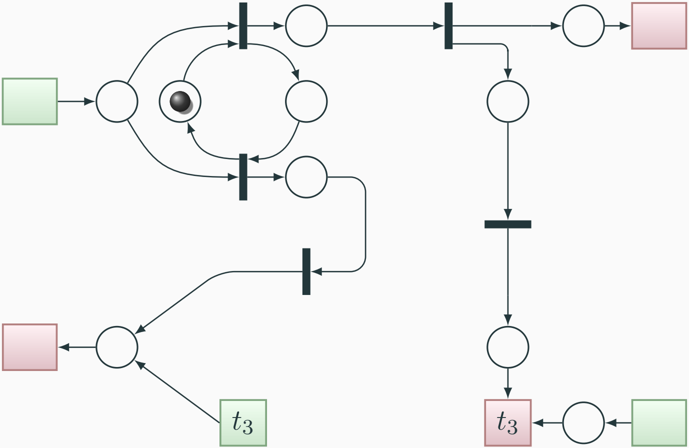

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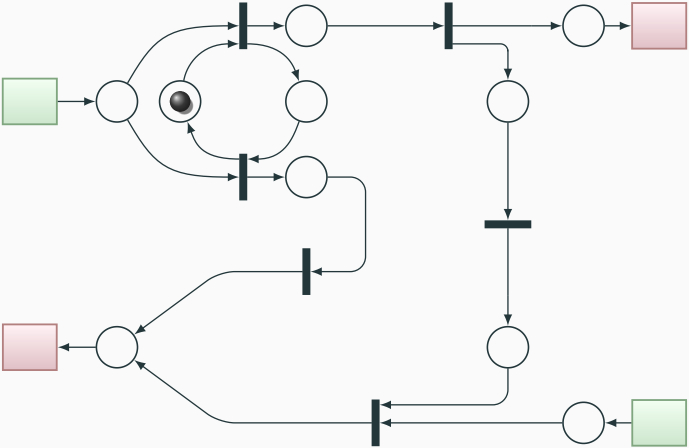

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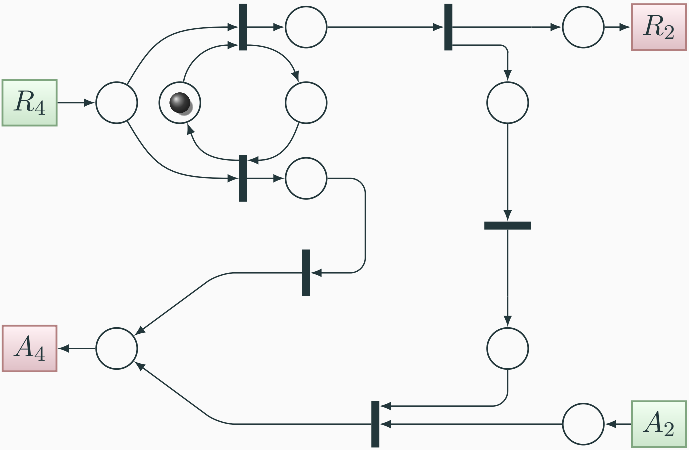

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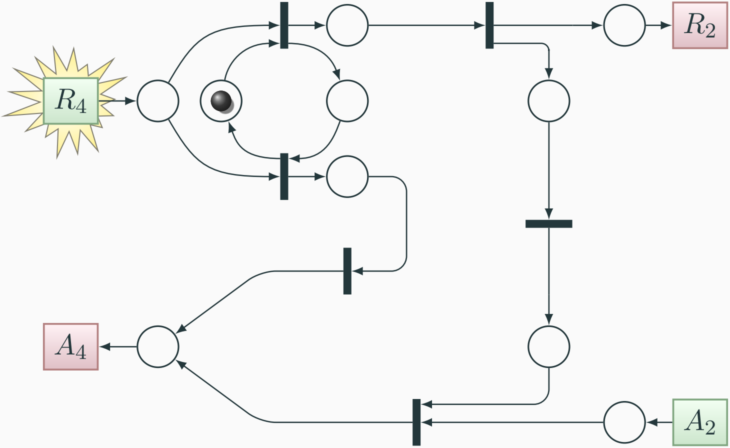

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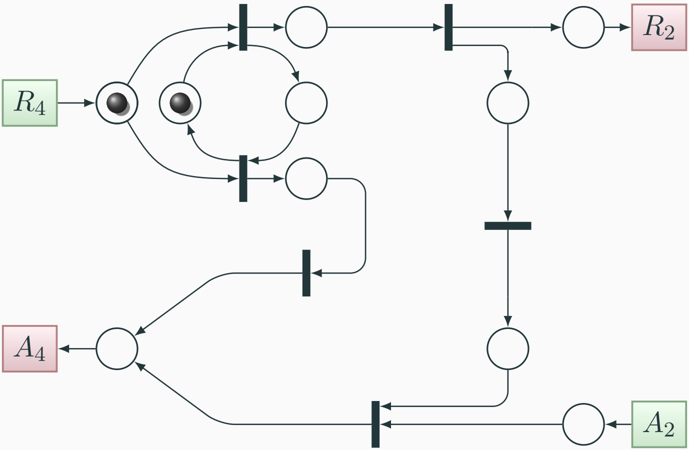

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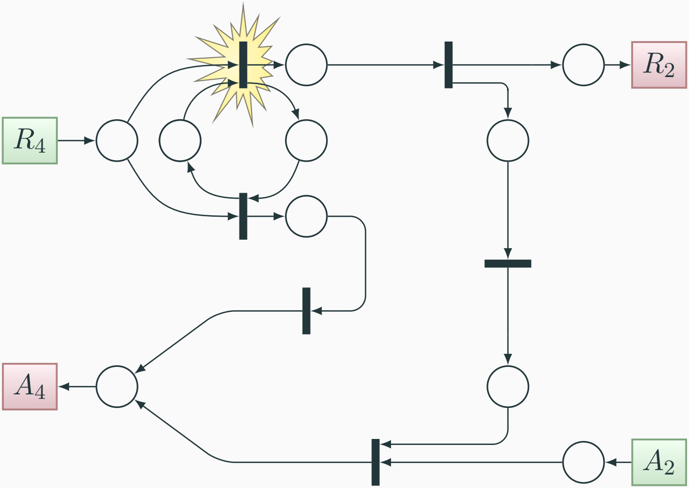

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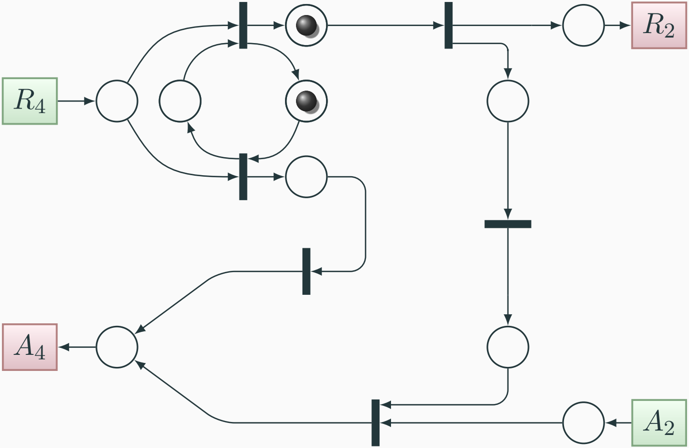

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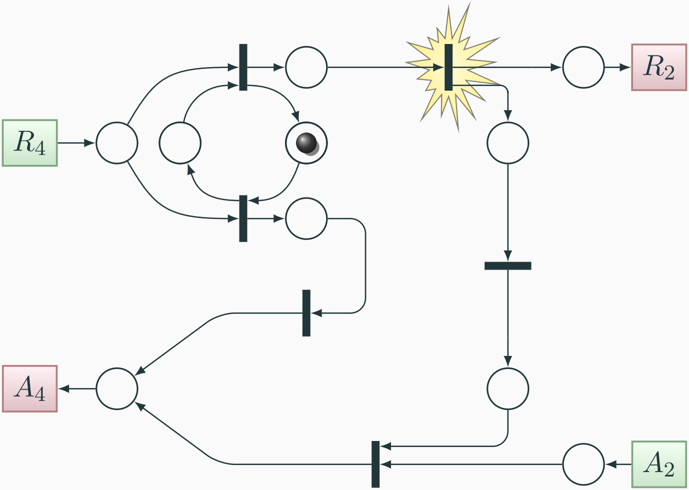

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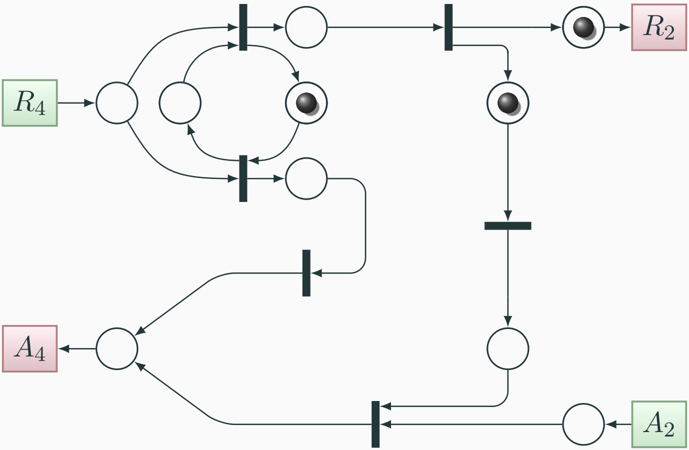

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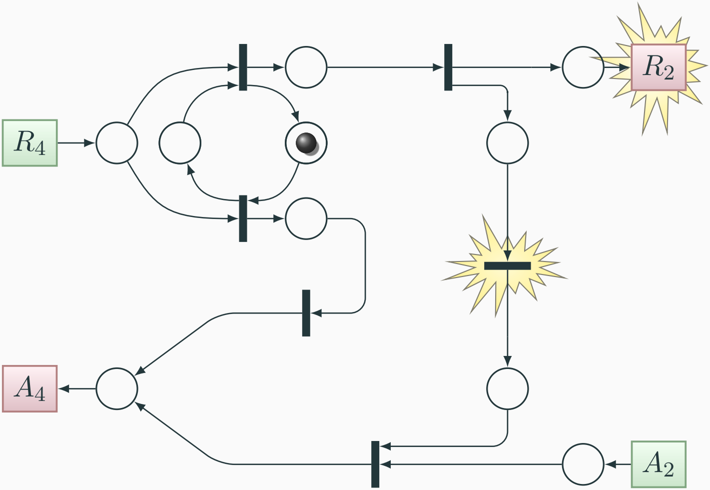

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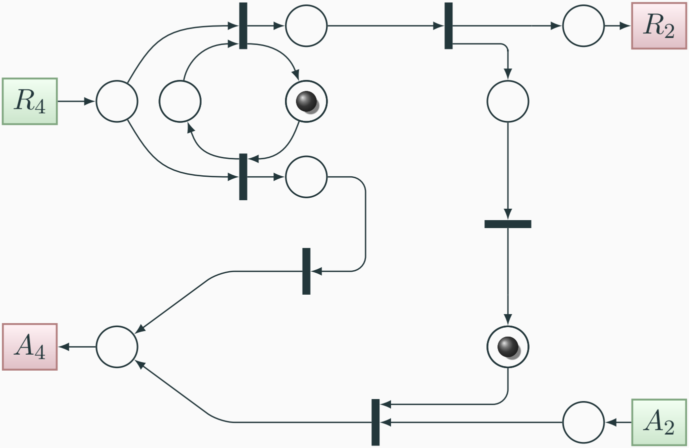

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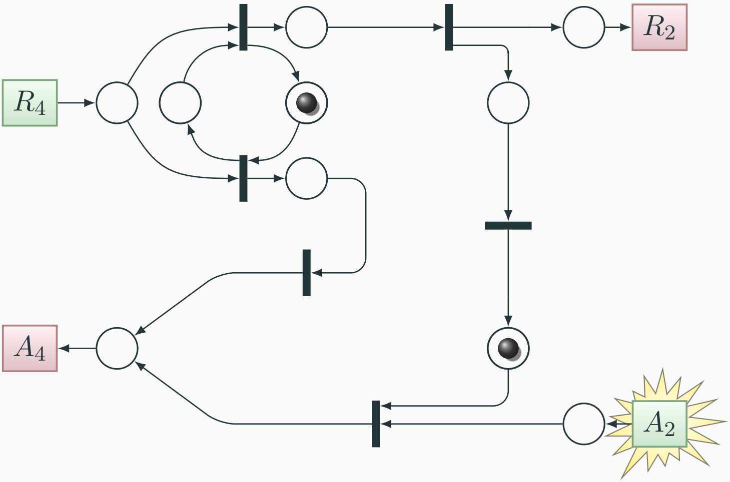

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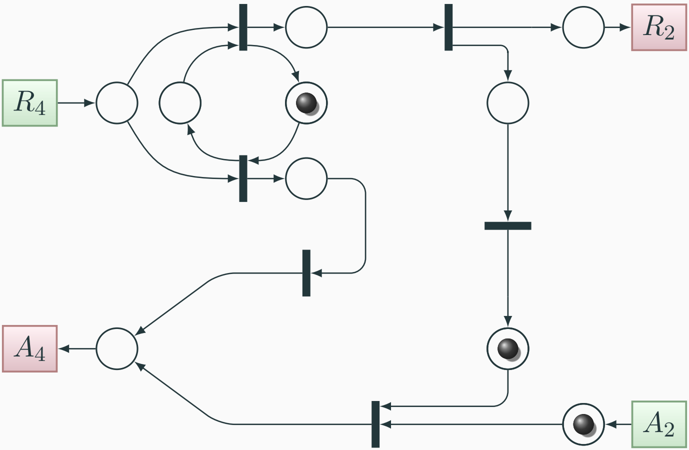

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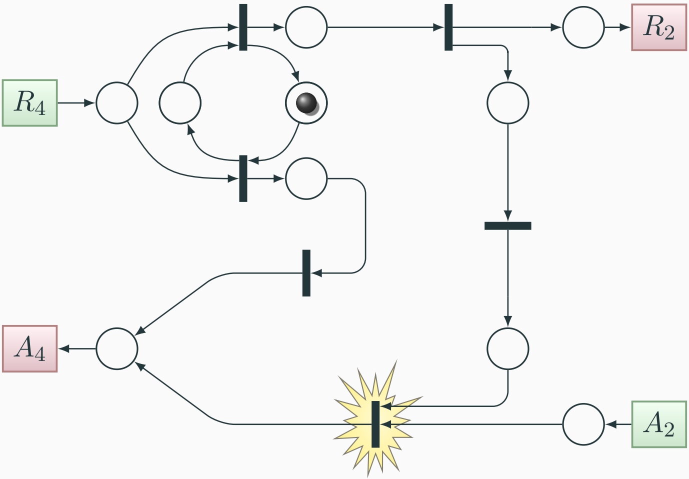

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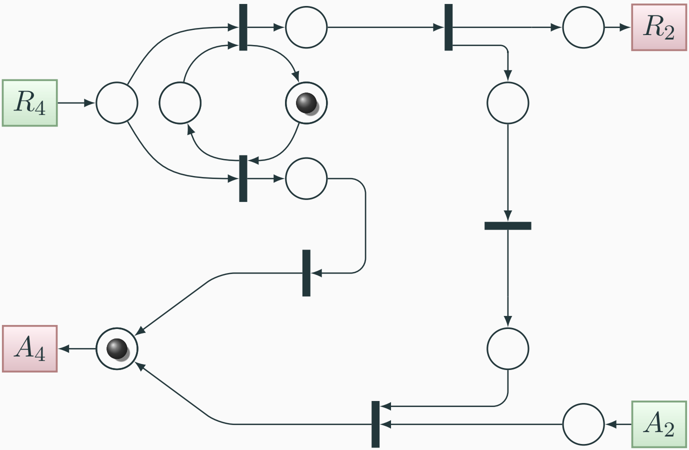

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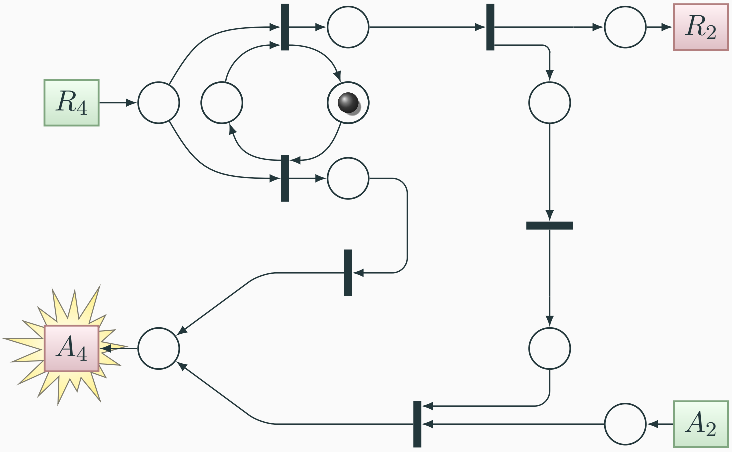

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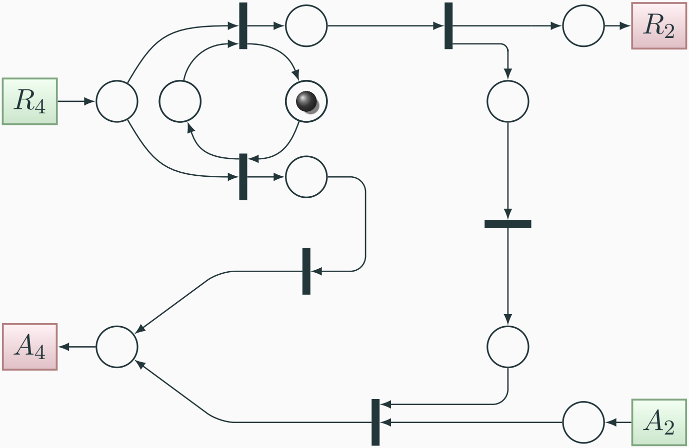

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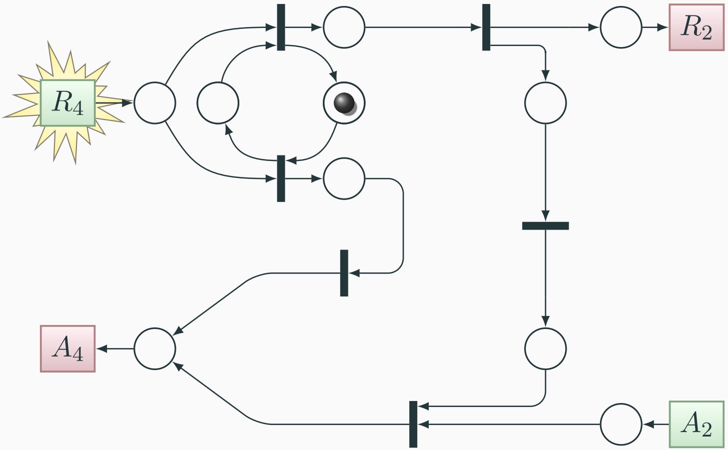

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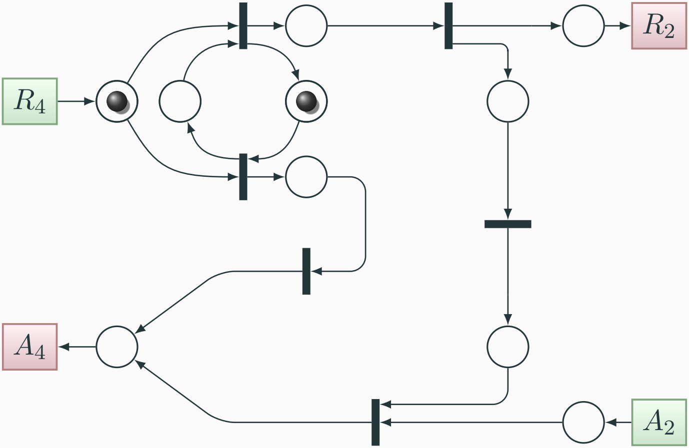

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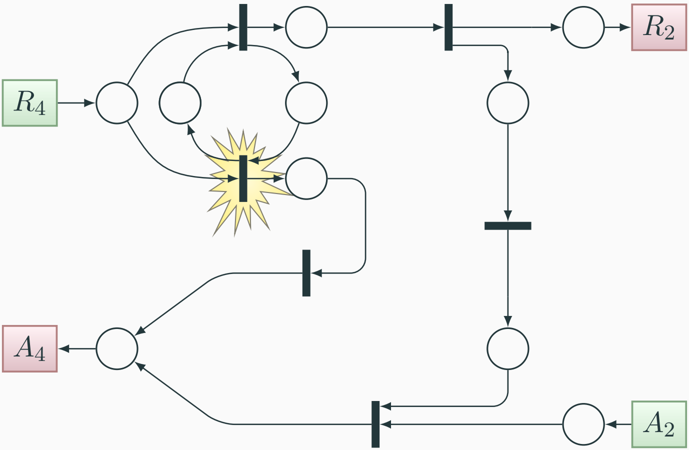

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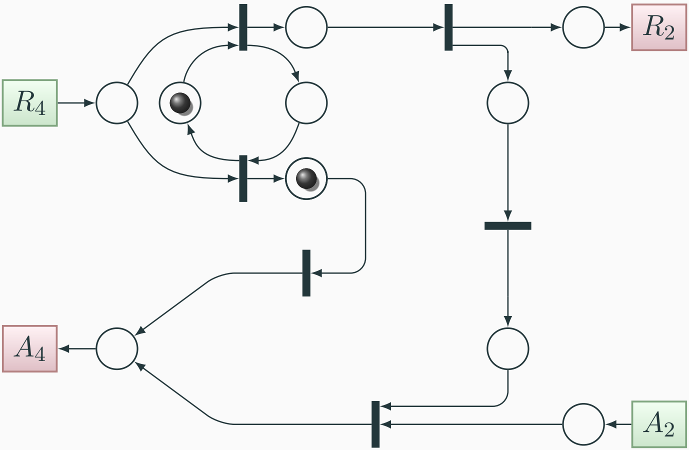

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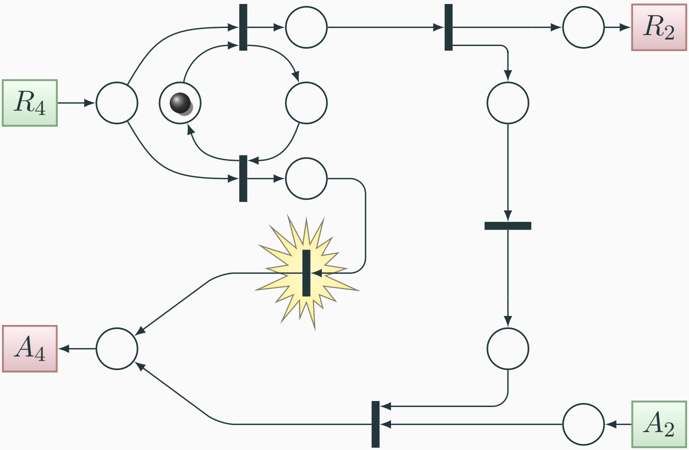

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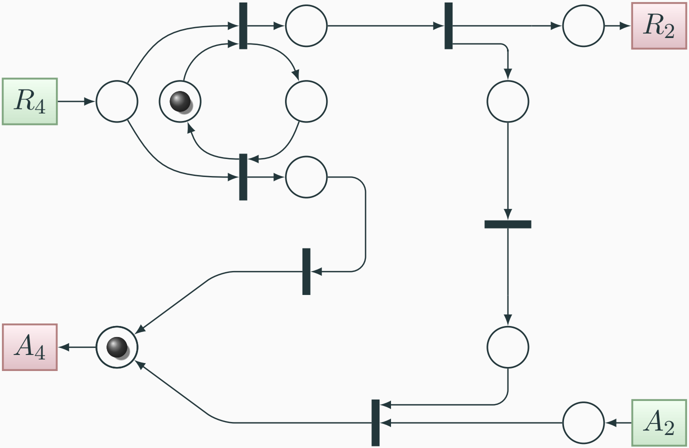

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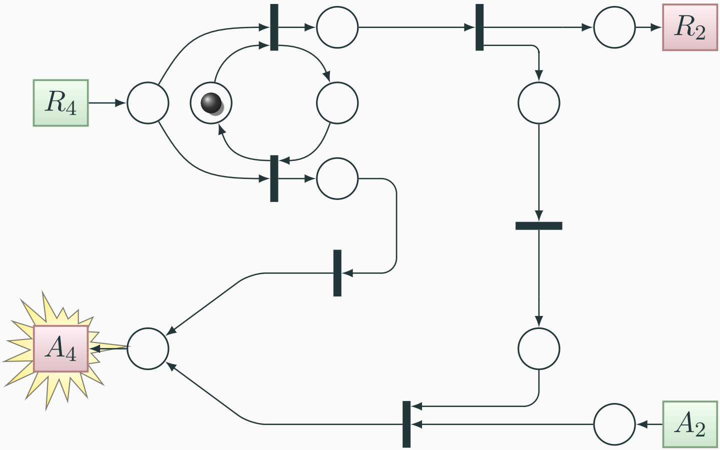

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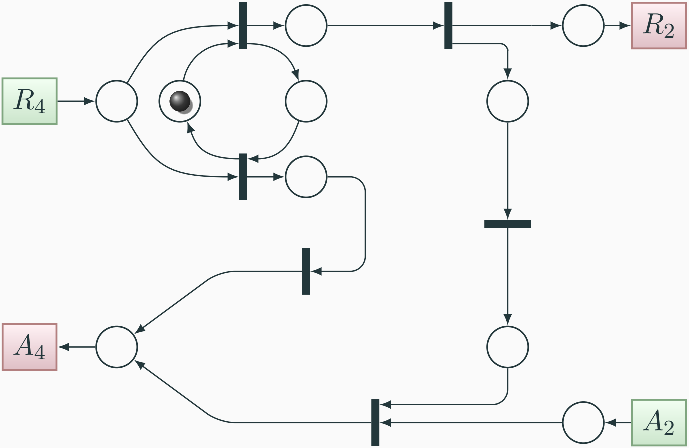

The formal approach

To-do list

Give a formal account of how components

- get connected into a network
- behave individually
- behave collectively when connected into a network

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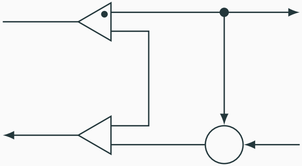
Block diagrams

Disregarding semantics, how can connections be specified ?



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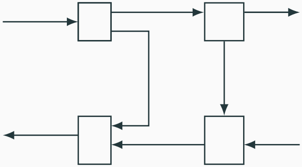

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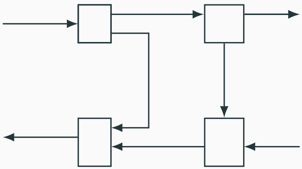
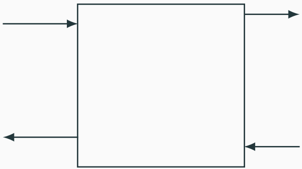


A formalism for block diagrams should enable

- algebraic description

Block diagrams

Disregarding semantics, how can connections be specified ?



A formalism for block diagrams should enable

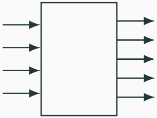
- algebraic description
- hierarchy

A *blockoid* over a set $\mathbf{b}$ is an algebraic structure $(\mathbf{b}, \mathbf{r}, \mathbf{z}, \mathbf{i})$ with

- $\mathbf{i} \in \mathbf{b}$
- $\mathbf{z} : \mathbf{b} \to \mathbf{b}$
- $\mathbf{r} : \mathbf{b} \times \mathbf{b} \to \mathbf{b}$

satisfying the *blockoid axioms*.

Intuition



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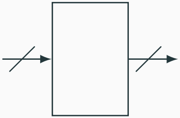

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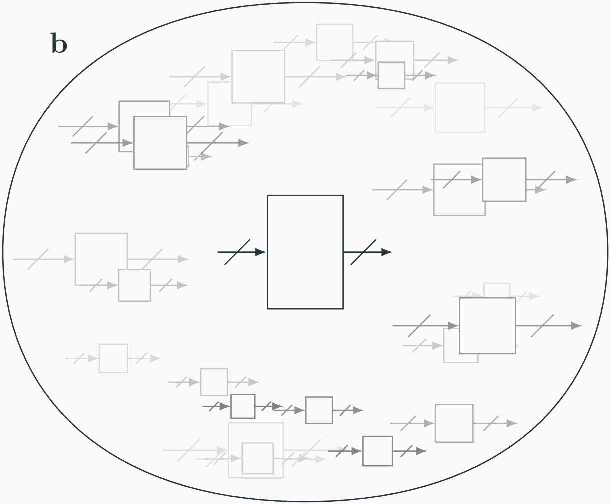
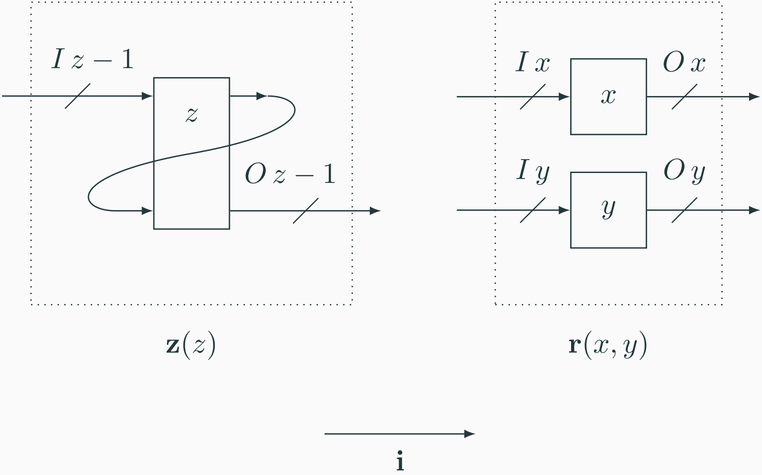
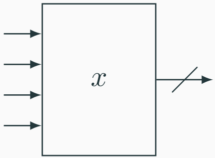


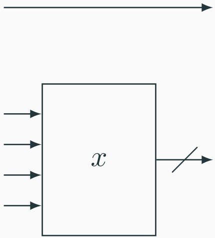
Intuition



Rolling down the inputs

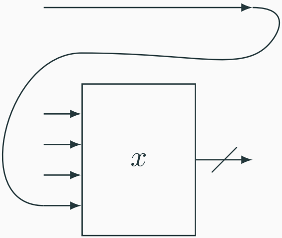


Rolling down the inputs



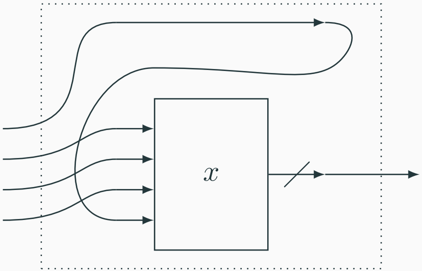
$$\mathbf{r}(\mathbf{i}, x)$$

Rolling down the inputs



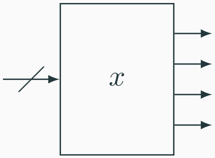
$$\mathbf{zr}(\mathbf{i}, x)$$

Rolling down the inputs

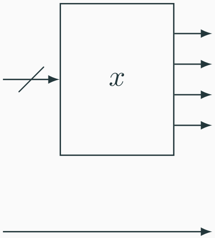


$$\mathbf{zr}(\mathbf{i}, x)$$

Rolling up the outputs

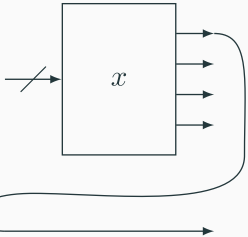


Rolling up the outputs



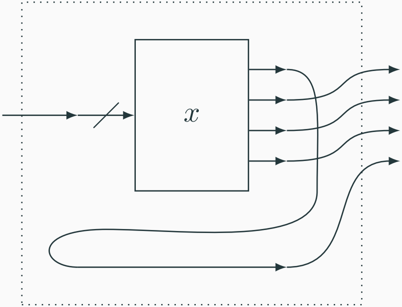
$$\mathbf{r}(x, \mathbf{i})$$

Rolling up the outputs



$$\mathbf{zr}(x, \mathbf{i})$$

Rolling up the outputs



$\mathbf{zr}(x, \mathbf{i})$

Blockoid axioms

There exists a *congruence* on $\mathbf{b}$ such that

- $\forall x \in \mathbf{b}.\ \exists i \in \mathbb{N}.\ \mathbf{t}\ x \equiv (\mathbf{z} \circ \mathbf{t})^{i+1}\, \mathbf{t}\ x$
- $\forall x \in \mathbf{b}.\ \exists o \in \mathbb{N}.\ \mathbf{v}\, x \equiv (\mathbf{z} \circ \mathbf{v})^{o+1}\, \mathbf{v}\, x$

are true, where $\mathbf{t}$ and $\mathbf{v}$ are defined by

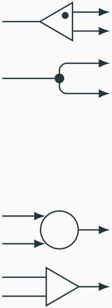
- $\mathbf{t} = \lambda x.\ \mathbf{r}(\mathbf{i}, x)$
- $\mathbf{v} = \lambda x.\ \mathbf{r}(x, \mathbf{i})$

The minimum $i, o \in \mathbb{N}$ satisfying these conditions for a block x are called its *input arity* $I\,x$ and its *output arity* $O\,x$.

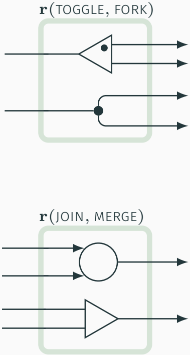
Blockoid axioms

idempotence	$\mathbf{zr}(\mathbf{i}, \mathbf{i}) \equiv \mathbf{i}$
left identity	$\mathbf{r}(\mathbf{z}\,\mathbf{i}, x) \equiv x$
right identity	$\mathbf{r}(x, \mathbf{z}\,\mathbf{i}) \equiv x$
associativity	$\mathbf{r}(x, \mathbf{r}(y, z)) \equiv \mathbf{r}(\mathbf{r}(x, y), z)$
input arity laws	$I\ \mathbf{z}\ x = I\ x - 1$
	$I\ \mathbf{r}(x, y) = I\ x + I\ y$
output arity laws	$O\ \mathbf{z}\ x = O\ x - 1$
	$O\ \mathbf{r}(x, y) = O\ y + O\ y$

Handshake converter blockoid

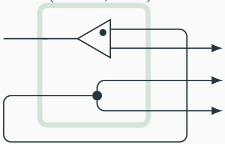


Handshake converter blockoid

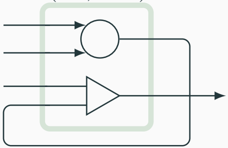


Handshake converter blockoid

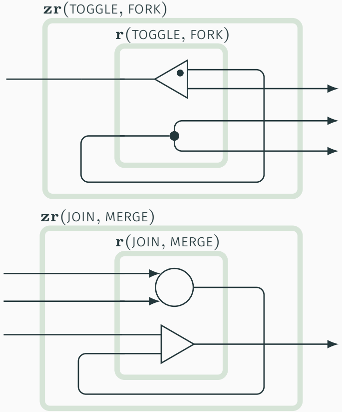
$\mathbf{r}$(TOGGLE, FORK)



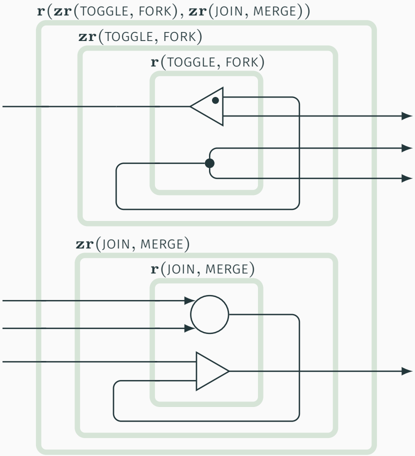
$\mathbf{r}$(JOIN, MERGE)



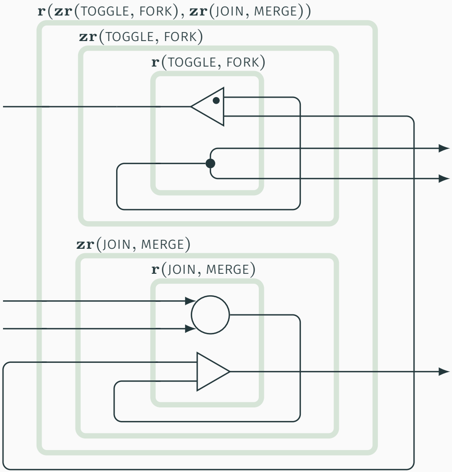
Handshake converter blockoid



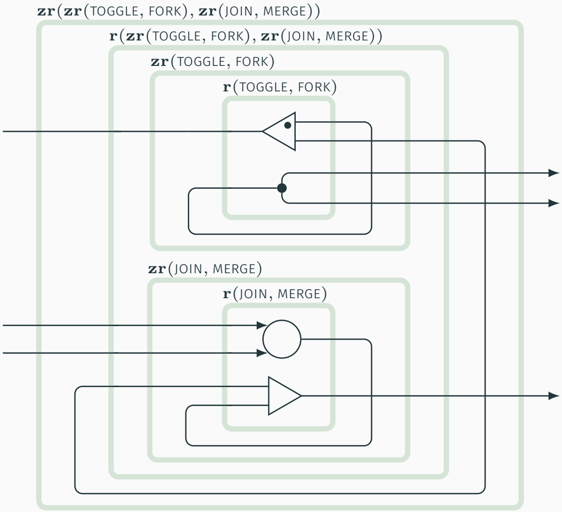
Handshake converter blockoid



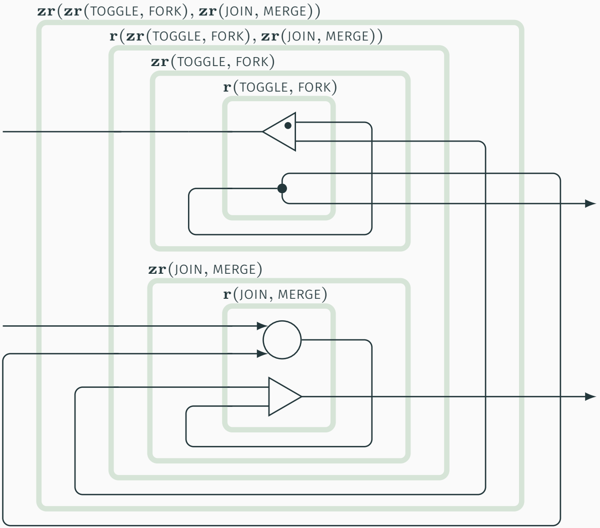
Handshake converter blockoid



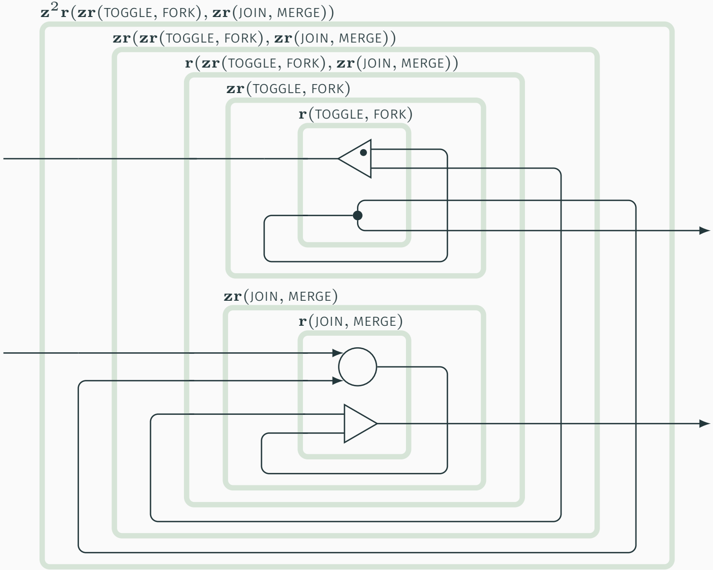
Handshake converter blockoid



Handshake converter blockoid

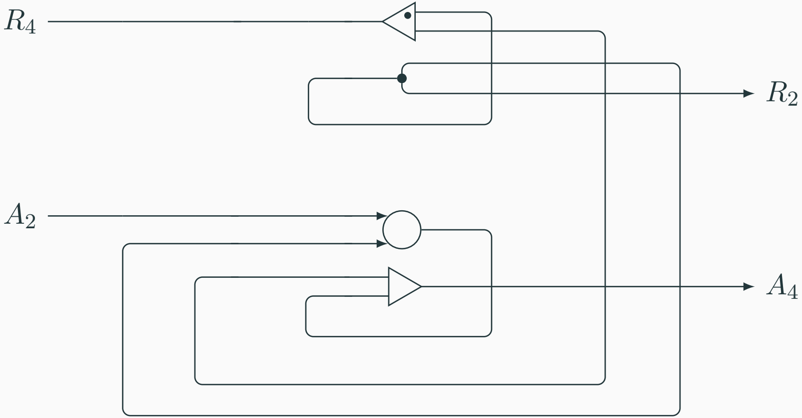


Handshake converter blockoid

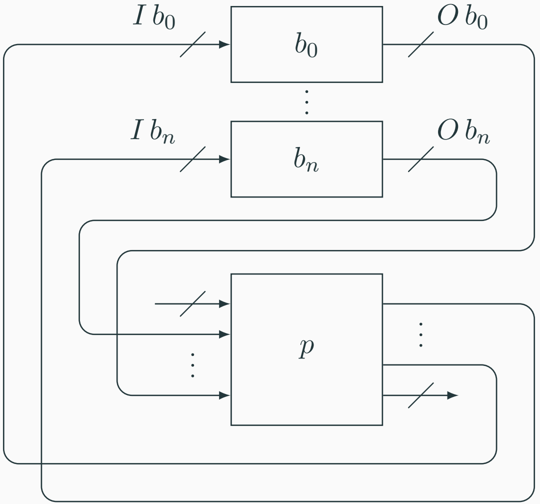


Handshake converter blockoid

$$\mathbf{z}^2\mathbf{r}(\mathbf{zr}(\text{TOGGLE}, \text{FORK}), \mathbf{zr}(\text{JOIN}, \text{MERGE}))$$



Universality of block combinators



Universality of block combinators

Any network of blocks $b = \langle b_0 \ldots b_n \rangle$ is expressible as

- bus from p to b
- exposed input bus
- bus from b to p

$$\mathbf{z}^w \mathbf{s}^v \mathbf{z}^u \, (F \; \mathbf{r}) \, (b \shortparallel \langle p \rangle)$$

$$\mathbf{t} = \lambda x.\, \mathbf{r}(\mathbf{i}, x) \qquad u = \sum_{t=0}^{n} O\, b_t$$

$$\mathbf{s} = \mathbf{z} \circ \mathbf{t} \qquad v = (I\, p) - u$$

$$w = \sum_{t=0}^{n} I\, b_t$$

for some *permutation network* p expressible by $\mathbf{r}$, $\mathbf{z}$ and $\mathbf{i}$.

Universality of block combinators

Any network of blocks $b = \langle b_0 \dots b_n \rangle$ is expressible as

- bus from p to b
- exposed input bus
- bus from b to p

$$\mathbf{z}^w \mathbf{s}^v \mathbf{z}^u \, (\digamma \, \mathbf{r}) \, (b \parallel \langle p \rangle)$$

fold

concatenate

$$\mathbf{t} = \lambda x.\ \mathbf{r}(\mathbf{i}, x) \qquad u = \sum_{t=0}^{n} O\, b_t$$

$$\mathbf{s} = \mathbf{z} \circ \mathbf{t} \qquad v = (I\, p) - u$$

$$w = \sum_{t=0}^{n} I\, b_t$$

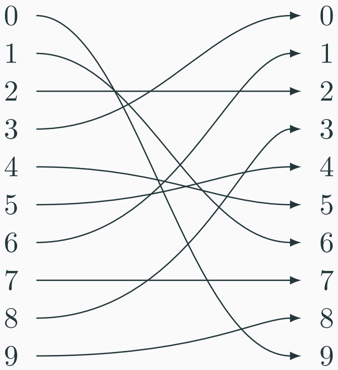
for some *permutation network* p expressible by $\mathbf{r}$, $\mathbf{z}$ and $\mathbf{i}$.

Permutation networks

x

$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$

$\mathbf{p}(x)$

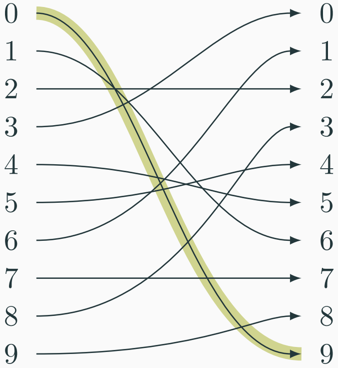


Permutation networks

x

$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$

$\mathbf{p}(x)$

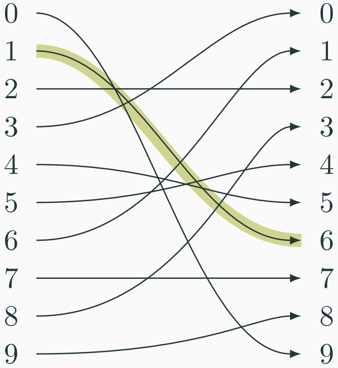


Permutation networks

x

$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$

$\mathbf{p}(x)$

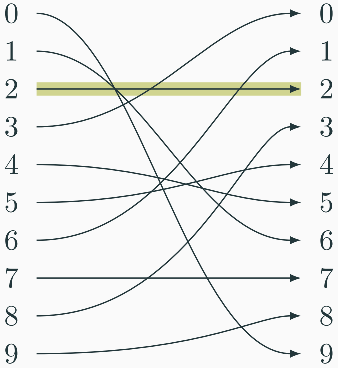


Permutation networks

x

$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$

$\mathbf{p}(x)$



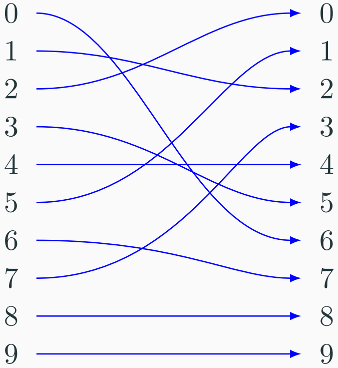
Permutation networks

x

$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$

$\langle 6, 2, 0, 5, 4, 1, 7, 3, 8, 9 \rangle$

$\mathbf{p}(x)$



Permutation networks

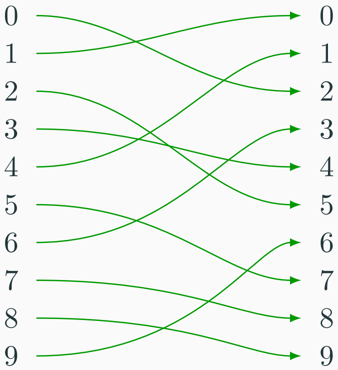
x

$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$

$\langle 6, 2, 0, 5, 4, 1, 7, 3, 8, 9 \rangle$

$\langle 2, 0, 5, 4, 1, 7, 3, 8, 9, 6 \rangle$

$\mathbf{p}(x)$



Permutation networks

x

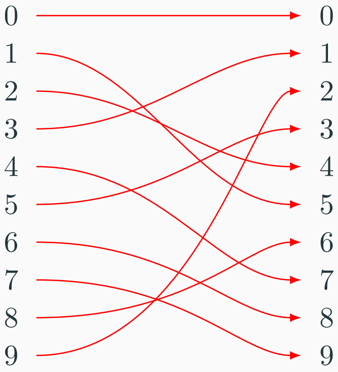
$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$

$\langle 6, 2, 0, 5, 4, 1, 7, 3, 8, 9 \rangle$

$\langle 2, 0, 5, 4, 1, 7, 3, 8, 9, 6 \rangle$

$\langle 0, 5, 4, 1, 7, 3, 8, 9, 6, 2 \rangle$

$\mathbf{p}(x)$



Permutation networks

$$x$$

$$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$$

$$\langle 6, 2, 0, 5, 4, 1, 7, 3, 8, 9 \rangle$$

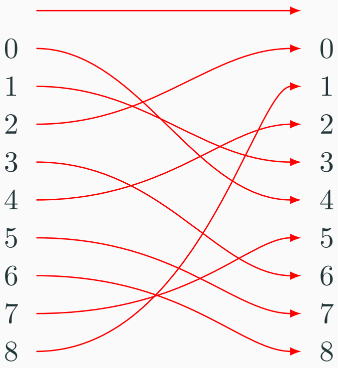
$$\langle 2, 0, 5, 4, 1, 7, 3, 8, 9, 6 \rangle$$

$$\langle \not{0}, 5, 4, 1, 7, 3, 8, 9, 6, 2 \rangle$$

$$\Downarrow -1$$

$$x' = \langle 4, 3, 0, 6, 2, 7, 8, 5, 1 \rangle$$

$$\mathbf{p}(x) = \mathbf{r}(\mathbf{i}, \mathbf{p}(x'))$$



Permutation networks

x

$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$

$\langle 6, 2, 0, 5, 4, 1, 7, 3, 8, 9 \rangle$

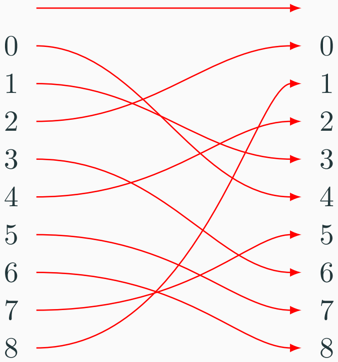
$\langle 2, 0, 5, 4, 1, 7, 3, 8, 9, 6 \rangle$

$\langle \cancel{0}, 5, 4, 1, 7, 3, 8, 9, 6, 2 \rangle$

$\Downarrow -1$

$$x' = \langle 4, 3, 0, 6, 2, 7, 8, 5, 1 \rangle$$
$$= (x \circlearrowleft h \ll 1) - 1$$

$\mathbf{p}(x) = \mathbf{r}(\mathbf{i}, \mathbf{p}(x'))$



Permutation networks

$$x$$

$$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8\rangle$$

$$\langle 6, 2, 0, 5, 4, 1, 7, 3, 8, 9\rangle$$

$$\langle 2, 0, 5, 4, 1, 7, 3, 8, 9, 6\rangle$$

$$\langle \not{0}, 5, 4, 1, 7, 3, 8, 9, 6, 2\rangle$$

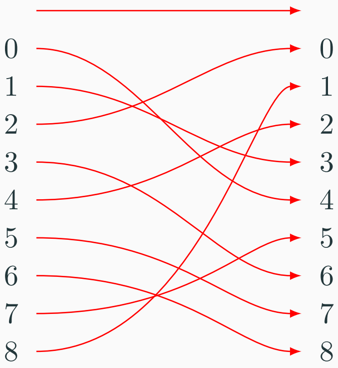
$$\Downarrow -1$$

$$x' = \langle 4, 3, 0, 6, 2, 7, 8, 5, 1\rangle$$

$$= (x \circlearrowleft h \ll 1) - 1$$

roll left *drop* *decrement*

$$\mathbf{p}(x) = \mathbf{r}(\mathbf{i}, \mathbf{p}(x'))$$



Permutation networks

$$x$$

$$\langle 9,6,2,0,5,4,1,7,3,8 \rangle$$

$$\langle 6,2,0,5,4,1,7,3,8,9 \rangle$$

$$\langle 2,0,5,4,1,7,3,8,9,6 \rangle$$

$$\langle \not{0},5,4,1,7,3,8,9,6,2 \rangle$$

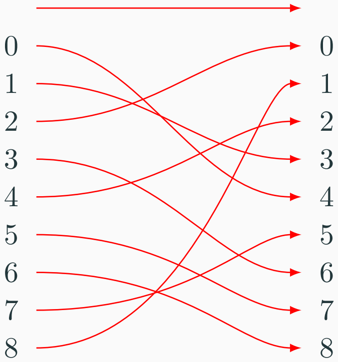
$$\Downarrow -1$$

$$x' = \langle 4,3,0,6,2,7,8,5,1 \rangle$$

$$= (x \circlearrowleft h \ll 1) - 1$$

where $h = (x^{-1})_0$

$$\mathbf{p}(x) = \mathbf{r}(\mathbf{i}, \mathbf{p}(x'))$$



Permutation networks

$$x$$

$$\langle 9,6,2,0,5,4,1,7,3,8\rangle$$

$$\langle 6,2,0,5,4,1,7,3,8,9\rangle$$

$$\langle 2,0,5,4,1,7,3,8,9,6\rangle$$

$$\langle \not{0},5,4,1,7,3,8,9,6,2\rangle$$

$$\Downarrow -1$$

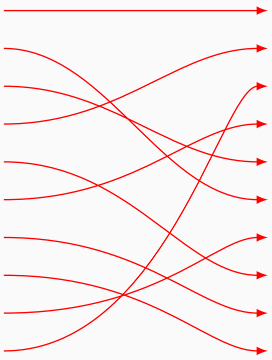
$$x' = \langle 4,3,0,6,2,7,8,5,1\rangle$$

$$= (x \circlearrowleft h \ll 1) - 1$$

where $h = (x^{-1})_0$

$$\mathbf{t} = \lambda a.\ \mathbf{r}(\mathbf{i}, a)$$

$$\mathbf{p}(x) = \mathbf{r}(\mathbf{i}, \mathbf{p}(x'))$$



Permutation networks

x

$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$

$\langle 6, 2, 0, 5, 4, 1, 7, 3, 8, 9 \rangle$

$\langle 2, 0, 5, 4, 1, 7, 3, 8, 9, 6 \rangle$

$\langle \cancel{0}, 5, 4, 1, 7, 3, 8, 9, 6, 2 \rangle$

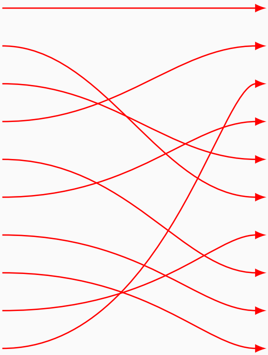
$\Downarrow -1$

$$x' = \langle 4, 3, 0, 6, 2, 7, 8, 5, 1 \rangle$$
$$= (x \circlearrowleft h \ll 1) - 1$$

where $h = (x^{-1})_0$

$\mathbf{t} = \lambda a.\ \mathbf{r}(\mathbf{i}, a)$

$\mathbf{p}(x) = \mathbf{tp}(x')$



Permutation networks

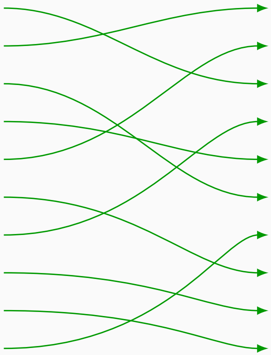
$$x$$

$$\langle 9,6,2,0,5,4,1,7,3,8\rangle$$
$$\langle 6,2,0,5,4,1,7,3,8,9\rangle$$
$$\langle 2,0,5,4,1,7,3,8,9,6\rangle$$
$$\langle \cancel{0},5,4,1,7,3,8,9,6,2\rangle$$
$$\Downarrow -1$$
$$x' = \langle 4,3,0,6,2,7,8,5,1\rangle$$
$$= (x \circlearrowleft h \ll 1) - 1$$

where $h = (x^{-1})_0$

$$\mathbf{t} = \lambda a.\ \mathbf{r}(\mathbf{i}, a)$$
$$\mathbf{s} = \mathbf{z} \circ \mathbf{t}$$

$$\mathbf{p}(x) = \mathbf{stp}(x')$$



Permutation networks

$$x$$

$$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$$

$$\langle 6, 2, 0, 5, 4, 1, 7, 3, 8, 9 \rangle$$

$$\langle 2, 0, 5, 4, 1, 7, 3, 8, 9, 6 \rangle$$

$$\langle \not{0}, 5, 4, 1, 7, 3, 8, 9, 6, 2 \rangle$$

$$\Downarrow -1$$

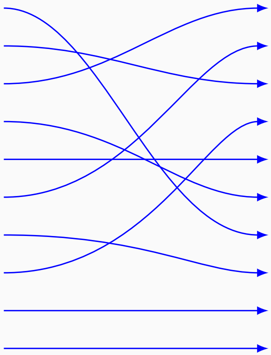
$$\begin{aligned} x' &= \langle 4, 3, 0, 6, 2, 7, 8, 5, 1 \rangle \\ &= (x \circlearrowleft h \ll 1) - 1 \end{aligned}$$

where $h = (x^{-1})_0$

$$\mathbf{t} = \lambda a.\ \mathbf{r}(\mathbf{i}, a)$$

$$\mathbf{s} = \mathbf{z} \circ \mathbf{t}$$

$$\mathbf{p}(x) = \mathbf{s}^2 \mathbf{t} \mathbf{p}(x')$$



Permutation networks

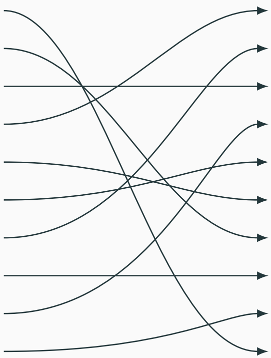
$$x$$

$$\langle 9, 6, 2, 0, 5, 4, 1, 7, 3, 8 \rangle$$
$$\langle 6, 2, 0, 5, 4, 1, 7, 3, 8, 9 \rangle$$
$$\langle 2, 0, 5, 4, 1, 7, 3, 8, 9, 6 \rangle$$
$$\langle \not{0}, 5, 4, 1, 7, 3, 8, 9, 6, 2 \rangle$$
$$\Downarrow -1$$
$$x' = \langle 4, 3, 0, 6, 2, 7, 8, 5, 1 \rangle$$
$$= (x \circlearrowleft h \ll 1) - 1$$

where $h = (x^{-1})_0$

$$\mathbf{t} = \lambda a.\ \mathbf{r}(\mathbf{i}, a)$$
$$\mathbf{s} = \mathbf{z} \circ \mathbf{t}$$

$$\mathbf{p}(x) = \mathbf{s}^h \mathbf{t} \mathbf{p}(x')$$



Permutation network combinator definition

The permutation obtained from $x \in \mathbb{N}^*$ rolled left by

$$h = (x^{-1})_0$$

decapitated and pointwise decremented

$$x' = (x \circlearrowleft h \ll 1) - 1$$

implies the recurrence

$$\mathbf{p}(x) = \begin{cases} \mathbf{i} & \text{if } |x| = 1 \\ (\lambda h.\, \mathbf{s}^h \mathbf{t} \mathbf{p}((x \circlearrowleft h \ll 1) - 1))\,(x^{-1})_0 & \text{if } |x| > 1 \end{cases}$$

implying universality of $\mathbf{r}$, $\mathbf{z}$, and $\mathbf{i}$ for block diagrams !

Permutation network examples

x	$\mathbf{p}(x)$
$\langle 9,6,2,0,5,4,1,7,3,8\rangle$	$\mathbf{s^3ts^2ts^5ts^3ts^4ts^4ts^3t^3i}$
$\langle 5,0,9,3,1,8,7,2,6,4\rangle$	$\mathbf{sts^2ts^2ts^4ts^3t^2s^3ts^2tsti}$
$\langle 4,5,1,7,3,2,8,0,9,6\rangle$	$\mathbf{s^7ts^4ts^2ts^6ts^3t^2s^3t^3i}$
$\langle 1,3,2,7,8,6,9,0,4,5\rangle$	$\mathbf{s^7ts^2tsts^6ts^4t^2s^2tst^2i}$
$\langle 2,9,6,0,1,5,3,7,8,4\rangle$	$\mathbf{s^3t^2s^5ts^3ts^2ts^2ts^3t^3i}$
$\langle 8,3,7,6,0,1,9,5,2,4\rangle$	$\mathbf{s^4t^2s^2ts^2ts^4ts^4ts^2ts^2tsti}$
$\langle 8,9,7,0,5,2,4,1,6,3\rangle$	$\mathbf{s^3ts^3ts^6ts^2ts^4ts^4t^2s^2t^2i}$
$\langle 0,9,2,7,1,4,3,8,5,6\rangle$	$\mathbf{ts^3ts^6ts^2ts^5tst^2st^2i}$
$\langle 5,3,4,8,1,7,0,6,2,9\rangle$	$\mathbf{s^6ts^7ts^2ts^2t^2s^4ts^2ts^2tsti}$
$\langle 0,6,2,4,8,5,3,7,9,1\rangle$	$\mathbf{ts^8tsts^3ts^3tsts^2tstsi}$
$\langle 3,0,7,9,1,8,6,2,5,4\rangle$	$\mathbf{sts^2ts^2ts^2ts^5ts^4ts^3t^2sti}$

To-do list

Give a formal account of how components

- ✓ get connected into a network
- behave individually
- behave collectively when connected into a network

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Component models – first attempt

Model the components as Petri nets.

$\mathbb{T}$	universe of observable transitions
$\mathbb{V}$	universe of places and unobservable transitions
$\mathbb{P}$	Petri nets (P, T, A, M, F)

- places $P \subset \mathbb{V}$
- transitions $T \subset \mathbb{T} \cup \mathbb{V}$
- arcs $A \subseteq (P \times T) \cup (T \times P)$
- initial marking $M \subseteq P$
- final marking $F \subseteq P$

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but then inputs and outputs are indistinguishable

Component models – second attempt

Model the components as DI processes.

$\mathbb{T}$	universe of observable transitions
$\mathbb{V}$	universe of places and unobservable transitions
$\mathbb{P}$	Petri nets (P, T, A, M, F)
$\mathbb{D}$	delay insensitive processes $p \in \mathscr{P}(\mathbb{T}) \times \mathscr{P}(\mathbb{T}) \times \mathbb{P}$

For $p = (I, O, N) \in \mathbb{D}$

- Petri net $N = (P, T, A, M, F) \in \mathbb{P}$
- input alphabet $I \supseteq T \cap \mathbb{T}$
- output alphabet $O \supseteq T \cap \mathbb{T}$

Refinement over processes

From a delay insensitive process $X = (I, O, N) \in \mathbb{D}$, infer

- a *reachability graph* $\mathbf{RG}(X)$ from the Petri net N.

From the reachability graph, infer

- a *quiescent trace recognizing automaton* $\mathbf{QR}(X)$
- a *divergent trace recognizing automaton* $\mathbf{DR}(X)$

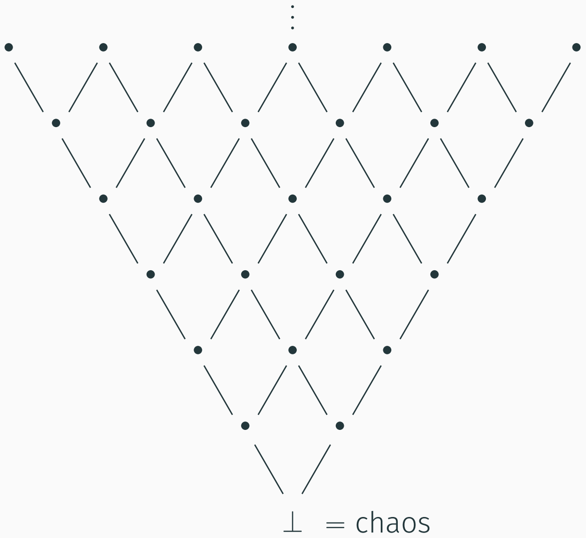
From their languages $\mathcal{L}\,\mathbf{QR}(X), \mathcal{L}\,\mathbf{DR}(X) \in (I \cup O)^*$, infer

- a *relational trace set* $[\![X]\!] = \mathcal{L}\,\mathbf{QR}(X) \cup \mathcal{L}\,\mathbf{DR}(X)$

such that the *refinement* relation $X \sqsubseteq Y$ coincides with

$$[\![X]\!] \supseteq [\![Y]\!].$$

Complete partial ordering by refinement



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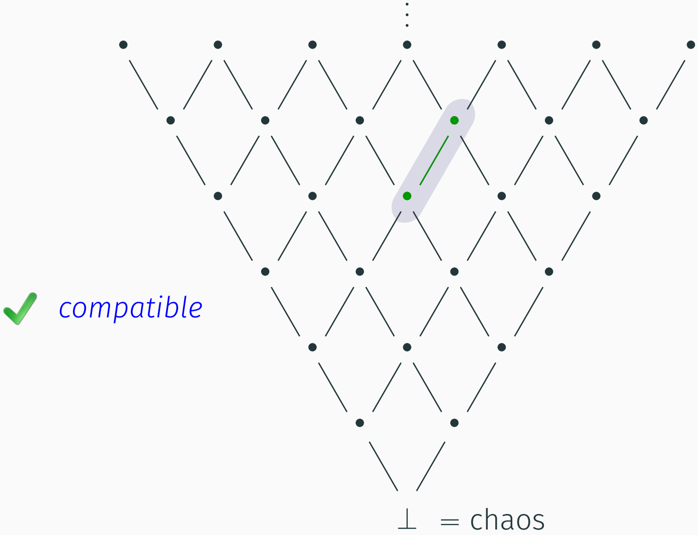

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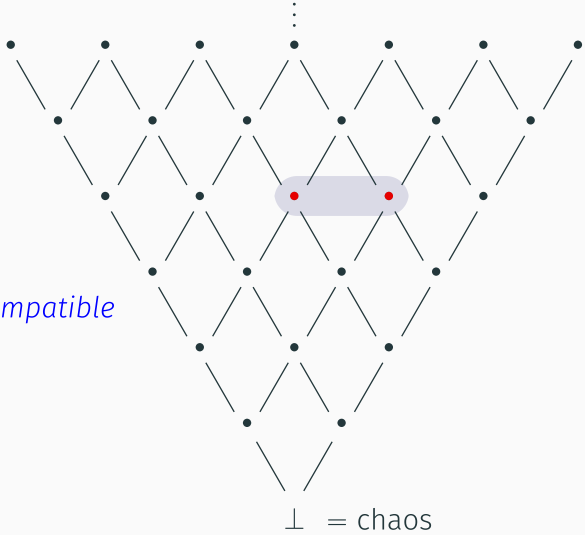

✖ *incompatible*

Component models – second attempt

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but then name clashes among transitions are inconvenient

Component models – third attempt

Model the components as blocks with terminals.

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$\mathbb{B}$	primitive blocks $b \in \mathbb{N} \times \mathbb{N} \times (((\mathbb{T}^\star \times \mathbb{T}^\star) \to \mathbb{D}))$

for $b = (I, O, B) \in \mathbb{B}$

- input arity $I \in \mathbb{N}$
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- process $(I', O', N) = B(i, o)$ has $I' = \mathscr{R}(i), O' = \mathscr{R}(o)$

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range of a list

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Refinement over components

For a standardized infinite list $\mathbb{G} \in \mathbb{T}^*$ of generic symbols, let

$$\mathcal{T}_{\mathbb{BD}} : \mathbb{B} \to \mathbb{D}$$

map components to processes by

$$\mathcal{T}_{\mathbb{BD}}(I, O, B) = B(\mathbb{G} \upharpoonright I, \mathbb{G} \ll I \upharpoonright O)$$

and let refinement among components $X, Y \in \mathbb{B}$ be defined by

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list truncation

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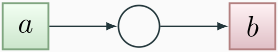
A blockoid $(\mathbb{B}, \mathbf{R}_{\mathbb{B}}, \mathbf{Z}_{\mathbb{B}}, \mathbf{I}_{\mathbb{B}})$ over circuit components

For the universe of primitive blocks

$$\mathbb{B} = \mathbb{N} \times \mathbb{N} \times (((\mathbb{T}^\star \times \mathbb{T}^\star) \to \mathbb{D}))$$

let $\mathbf{I}_{\mathbb{B}} = (I, O, B_{\mathbf{I}}) \in \mathbb{B}$ have $I = O = 1$ and

$$B_{\mathbf{I}}(\langle a \rangle, \langle b \rangle) = (\{a\}, \{b\}, N)$$



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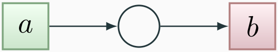

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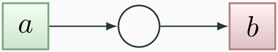
for some arbitrary but fixed place p.



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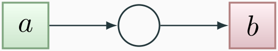

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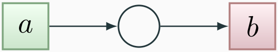

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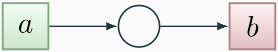

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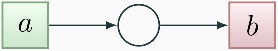

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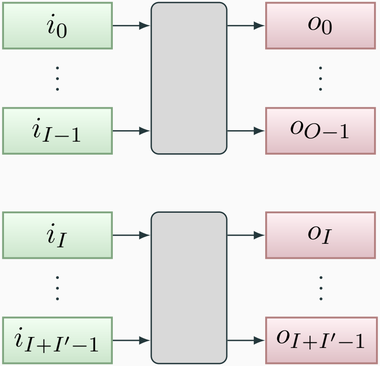
$$B_\mathbf{R}(i, o) = \mathsf{par}\ (B(i \shortmid I, o \shortmid O), B'(i \ll I, o \ll O))$$



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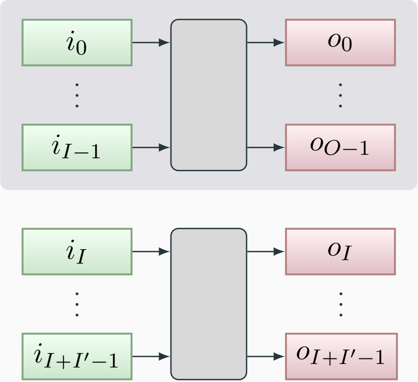

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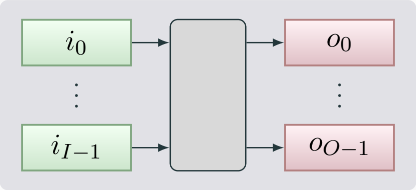
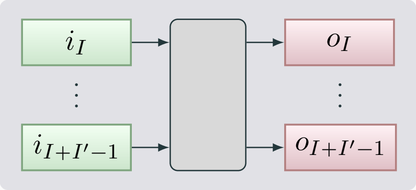


$B'(i \ll I, o \ll I)$



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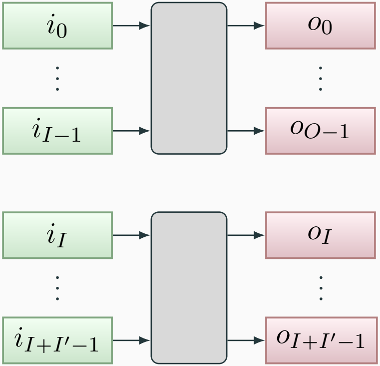
Parallel composition of DI processes



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Let $\mathbf{Z}_{\mathbb{B}}(I, O, B) = (I - 1, O - 1, B_{\mathbf{Z}}) \in \mathbb{B}$ with

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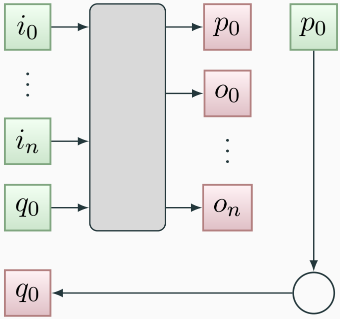
for arbitrary distinct $p, q \in \mathbb{T}^1$ disjoint from i and o.



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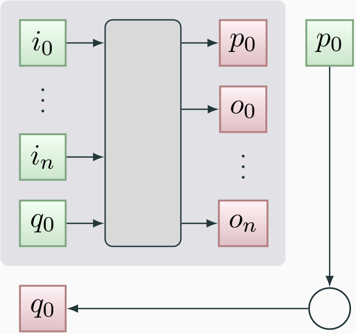

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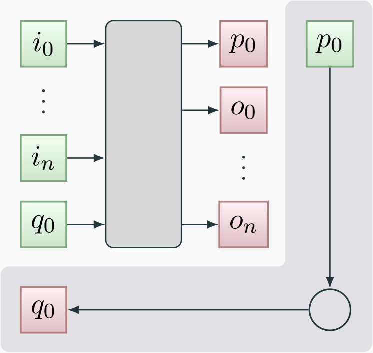

$B_{\mathbf{I}}(p, q)$

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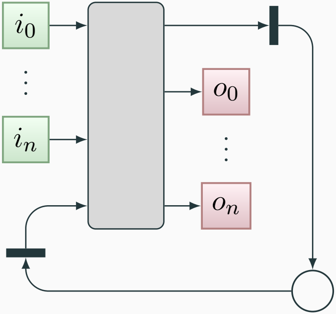

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Now what ?

- A network needs to be physically implemented somehow.
- Technology mapping tools need it in netlist form.
- How to ensure the netlist matches the block expression ?

An abstract representation

Idea !

Use an intermediate source with two possible targets.

- Let $\mathbb{H}$ denote a universe of *hierarchical blocks*.
- Let $\mathbb{L}$ denote a universe of netlists.
- Transform $\mathbb{B} \xleftarrow{\mathcal{T}_{\mathbb{HB}}} \mathbb{H} \xrightarrow{\mathcal{T}_{\mathbb{HL}}} \mathbb{L}$ in either direction.
- Make the transformations $\mathcal{T}_{\mathbb{HB}}$ and $\mathcal{T}_{\mathbb{HL}}$ simple and obvious.

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Structure-preserving maps between blockoids over $\mathbb{H}$, $\mathbb{B}$, and $\mathbb{L}$?

A blockoid $(\mathbb{H}, \mathbf{R}, \mathbf{Z}, \mathbf{I})$ over hierarchical blocks

Define the universe of hierarchical blocks

$$\mathbb{H} = \mathbb{B} \cup \mathbb{H}^*$$

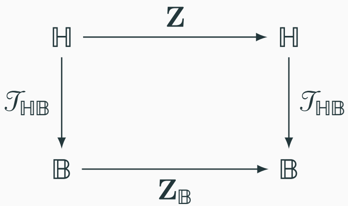
and let nested lists encode them by block combinators

$$\begin{array}{lcl}
\mathbf{I} & = & \mathbf{I}_{\mathbb{B}} \\
\mathbf{Z}(x) & = & \langle x \rangle \\
\mathbf{R}(\langle x \rangle, \langle y \rangle) & = & \langle \langle x \rangle, \langle y \rangle \rangle \\
\mathbf{R}(\langle x \rangle, Y) & = & \langle \langle x \rangle \rangle \shortparallel Y \\
\mathbf{R}(X, \langle y \rangle) & = & X \shortparallel \langle \langle y \rangle \rangle \\
\mathbf{R}(X, Y) & = & X \shortparallel Y
\end{array}$$

for all $x, y \in \mathbb{H}$ and $X, Y \in \mathbb{H}^*$ with $|X|, |Y| > 1$.

Transformation from hierarchical to primitive blocks

A transformation $\mathcal{T}_{\mathbb{HB}} : \mathbb{H} \to \mathbb{B}$ satisfying



determines a refinement relation

$$X \overset{\alpha}{\sqsubseteq} Y \Leftrightarrow \mathcal{T}_{\mathbb{HB}}\, X \overset{\epsilon}{\sqsubseteq} \mathcal{T}_{\mathbb{HB}}\, Y$$

and hence an extensional semantics for all $X, Y \in \mathbb{H}$.

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$$
\begin{array}{ccc}
\mathbb{H} \times \mathbb{H} & \xrightarrow{\quad \mathbf{R} \quad} & \mathbb{H} \\
{\scriptstyle \lambda(x,y).\ (\mathcal{T}_{\mathbb{HB}}\, x,\, \mathcal{T}_{\mathbb{HB}}\, y)} \big\downarrow & & \big\downarrow {\scriptstyle \mathcal{T}_{\mathbb{HB}}} \\
\mathbb{B} \times \mathbb{B} & \xrightarrow[\quad \mathbf{R}_{\mathbb{B}} \quad]{} & \mathbb{B}
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$$

determines a refinement relation

$$X \overset{\alpha}{\sqsubseteq} Y \Leftrightarrow \mathcal{T}_{\mathbb{HB}}\, X \overset{\epsilon}{\sqsubseteq} \mathcal{T}_{\mathbb{HB}}\, Y$$

and hence an extensional semantics for all $X, Y \in \mathbb{H}$.

Transformation from hierarchical to primitive blocks

A transformation $\mathscr{T}_{\mathbb{HB}} : \mathbb{H} \to \mathbb{B}$ satisfying

$$\mathscr{T}_{\mathbb{HB}}(h) = \begin{cases} h & \text{if } h \in \mathbb{B} \\ \mathbf{Z}_{\mathbb{B}}\, \mathscr{T}_{\mathbb{HB}}\, h_0 & \text{if } h \in \mathbb{H}^1 \\ (F\ \mathbf{R}_{\mathbb{B}})\, \mathscr{T}^{\star}_{\mathbb{HB}}\, h & \text{otherwise} \end{cases}$$

determines a refinement relation

$$X \overset{\alpha}{\sqsubseteq} Y \Leftrightarrow \mathscr{T}_{\mathbb{HB}}\, X \overset{\epsilon}{\sqsubseteq} \mathscr{T}_{\mathbb{HB}}\, Y$$

and hence an extensional semantics for all $X, Y \in \mathbb{H}$.

Transformation from hierarchical to primitive blocks

A transformation $\mathcal{T}_{\mathbb{HB}} : \mathbb{H} \to \mathbb{B}$ satisfying

map over a list

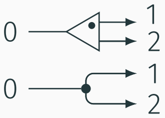
$$\mathcal{T}_{\mathbb{HB}}(h) = \begin{cases} h & \text{if } h \in \mathbb{B} \\ \mathbf{Z}_{\mathbb{B}}\, \mathcal{T}_{\mathbb{HB}}\, h_0 & \text{if } h \in \mathbb{H}^1 \\ (F\ \mathbf{R}_{\mathbb{B}})\, \mathcal{T}^{\star}_{\mathbb{HB}}\, h & \text{otherwise} \end{cases}$$

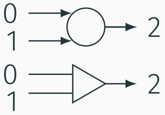
determines a refinement relation

$$X \overset{\alpha}{\sqsubseteq} Y \Leftrightarrow \mathcal{T}_{\mathbb{HB}}\, X \overset{\epsilon}{\sqsubseteq} \mathcal{T}_{\mathbb{HB}}\, Y$$

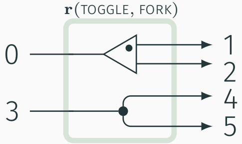
and hence an extensional semantics for all $X, Y \in \mathbb{H}$.

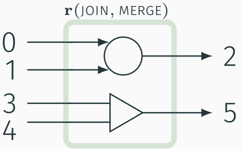
Netlists from blockoid operators





Netlists from blockoid operators





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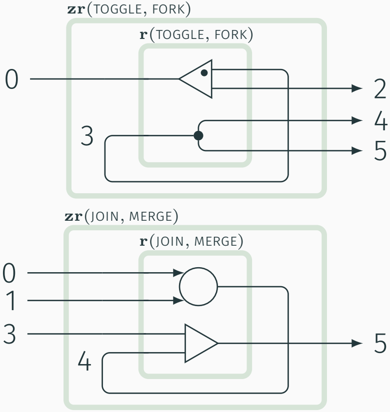
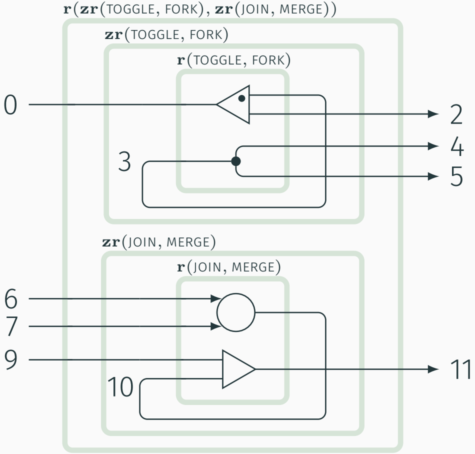


Netlists from blockoid operators



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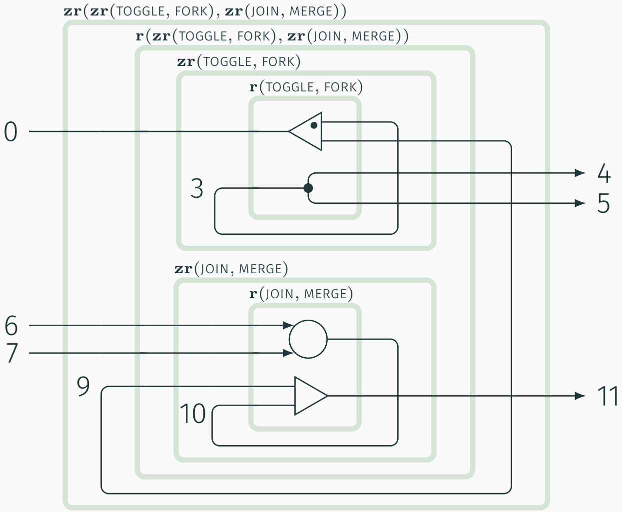
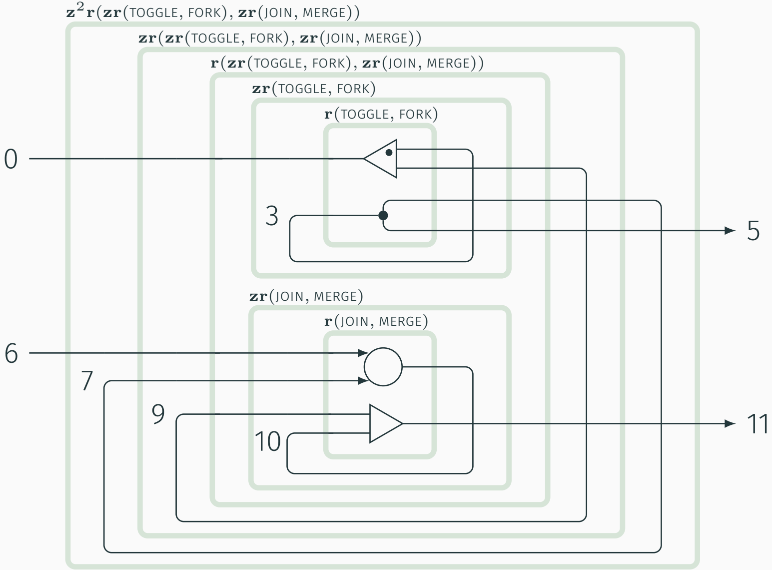
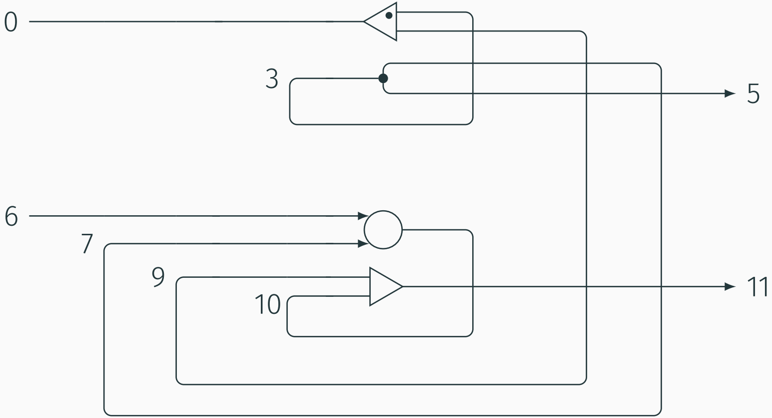
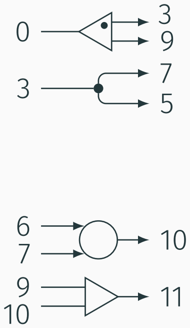


Netlists from blockoid operators

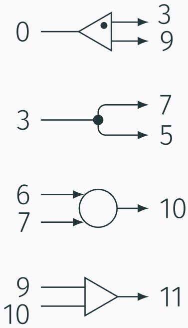



0
3
5
6
7
9
10
11

Netlists from blockoid operators



Netlists from blockoid operators



$(\langle 0 \rangle, \langle 3, 9 \rangle, \textsc{toggle})$

$(\langle 3 \rangle, \langle 7, 5 \rangle, \textsc{fork})$

$(\langle 6, 7 \rangle, \langle 10 \rangle, \textsc{join})$

$(\langle 9, 10 \rangle, \langle 11 \rangle, \textsc{merge})$

Netlists from blockoid operators

Netlists boil down to members of $(\mathbb{N}^* \times \mathbb{N}^* \times \mathbb{B})^*$ such as

$$\begin{aligned}\langle\quad & \\ & (\langle 0\rangle, && \langle 3,9\rangle, && \textsc{toggle}), \\ & (\langle 3\rangle, && \langle 7,5\rangle, && \textsc{fork}), \\ & (\langle 6,7\rangle, && \langle 10\rangle, && \textsc{join}), \\ & (\langle 9,10\rangle, && \langle 11\rangle, && \textsc{merge})\rangle\end{aligned}$$

A blockoid $(\mathbb{L}, \mathbf{R}_{\mathbb{L}}, \mathbf{Z}_{\mathbb{L}}, \mathbf{I}_{\mathbb{L}})$ over netlists

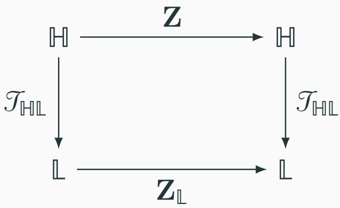
Define the universe of netlists

$$\mathbb{L} = (\mathbb{N}^* \times \mathbb{N}^* \times \mathbb{B})^*$$

with

$$\mathbf{I}_{\mathbb{L}} = \big\langle (\langle 0 \rangle, \langle 1 \rangle, \mathbf{I}_{\mathbb{B}}) \big\rangle$$

and $\mathbf{Z}_{\mathbb{L}}, \mathbf{R}_{\mathbb{L}}$ satisfying



A blockoid $(\mathbb{L}, \mathbf{R}_{\mathbb{L}}, \mathbf{Z}_{\mathbb{L}}, \mathbf{I}_{\mathbb{L}})$ over netlists

Define the universe of netlists

$$\mathbb{L} = (\mathbb{N}^* \times \mathbb{N}^* \times \mathbb{B})^*$$

with

$$\mathbf{I}_{\mathbb{L}} = \Big\langle \big(\langle 0 \rangle, \langle 1 \rangle, \mathbf{I}_{\mathbb{B}} \big) \Big\rangle$$

and $\mathbf{Z}_{\mathbb{L}}, \mathbf{R}_{\mathbb{L}}$ satisfying

$$\begin{array}{ccc}
\mathbb{H} \times \mathbb{H} & \xrightarrow{\quad \mathbf{R} \quad} & \mathbb{H} \\
{\scriptstyle \lambda(x,y).\ (\mathscr{T}_{\mathbb{H}\mathbb{L}}\, x, \mathscr{T}_{\mathbb{H}\mathbb{L}}\, y)} \downarrow & & \downarrow {\scriptstyle \mathscr{T}_{\mathbb{H}\mathbb{L}}} \\
\mathbb{L} \times \mathbb{L} & \xrightarrow[\quad \mathbf{R}_{\mathbb{L}} \quad]{} & \mathbb{L}
\end{array}$$

A blockoid $(\mathbb{L}, \mathbf{R}_{\mathbb{L}}, \mathbf{Z}_{\mathbb{L}}, \mathbf{I}_{\mathbb{L}})$ over netlists

Define the universe of netlists

$$\mathbb{L} = (\mathbb{N}^* \times \mathbb{N}^* \times \mathbb{B})^*$$

with

$$\mathbf{I}_{\mathbb{L}} = \left\langle (\langle 0 \rangle, \langle 1 \rangle, \mathbf{I}_{\mathbb{B}}) \right\rangle$$

and $\mathbf{Z}_{\mathbb{L}}, \mathbf{R}_{\mathbb{L}}$ satisfying

$$\mathcal{T}_{\mathbb{HL}}(h) = \begin{cases} (\lambda(I, O, B).\, \langle (\iota_I, \iota_O^I, h) \rangle)\ h & \text{if } h \in \mathbb{B} \\ \mathbf{Z}_{\mathbb{L}}\, \mathcal{T}_{\mathbb{HL}}\, h_0 & \text{if } h \in \mathbb{H}^1 \\ (\digamma\, \mathbf{R}_{\mathbb{L}})\, \mathcal{T}^{\star}_{\mathbb{HL}}\, h & \text{otherwise} \end{cases}$$

Define the universe of netlists

$$\mathbb{L} = (\mathbb{N}^* \times \mathbb{N}^* \times \mathbb{B})^*$$

with

$$\mathbf{I}_{\mathbb{L}} = \big\langle (\langle 0 \rangle, \langle 1 \rangle, \mathbf{I}_{\mathbb{B}}) \big\rangle$$

and $\mathbf{Z}_{\mathbb{L}}, \mathbf{R}_{\mathbb{L}}$ satisfying

$\langle 0 \ldots I - 1 \rangle$

$\langle I \ldots I + O - 1 \rangle$

$$\mathcal{T}_{\mathbb{HL}}(h) = \begin{cases} (\lambda(I, O, B).\, \langle(\iota_I, \iota_O^I, h)\rangle)\ h & \text{if } h \in \mathbb{B} \\ \mathbf{Z}_{\mathbb{L}}\ \mathcal{T}_{\mathbb{HL}}\ h_0 & \text{if } h \in \mathbb{H}^1 \\ (F\ \mathbf{R}_{\mathbb{L}})\ \mathcal{T}^{\star}_{\mathbb{HL}}\ h & \text{otherwise} \end{cases}$$

A blockoid $(\mathbb{L}, \mathbf{R}_\mathbb{L}, \mathbf{Z}_\mathbb{L}, \mathbf{I}_\mathbb{L})$ over netlists

For input terminal numbers and output terminal numbers

$$i(x) = \bigcup_{(I,O,B) \in \mathscr{R}(x)} \mathscr{R}(I) \qquad o(x) = \bigcup_{(I,O,B) \in \mathscr{R}(x)} \mathscr{R}(O)$$

associated with a netlist $x \in \mathbb{L}$, we have

all terminals	$i(x) \cup o(x)$
all external inputs	$i(x) - o(x)$
all external outputs	$o(x) - i(x)$
first external output	$\min\big(o(x) - i(x)\big)$
last external input	$\max\big(i(x) - o(x)\big)$

A rewrite rule for input terminal numbers

$$w_z(x) = \lambda t.\ \begin{cases} \min(o(x) - i(x)) & \text{if } t = \max(i(x) - o(x)) \\ t & \text{otherwise} \end{cases}$$

specifies the operator $\mathbf{Z}_\mathbb{L} : \mathbb{L} \to \mathbb{L}$

$$\mathbf{Z}_\mathbb{L}(x) = (\lambda(I, O, B).\ ((w_z\, x)^\star\, I, O, B))^\star\, x$$

A blockoid $(\mathbb{L}, \mathbf{R}_{\mathbb{L}}, \mathbf{Z}_{\mathbb{L}}, \mathbf{I}_{\mathbb{L}})$ over netlists

A rewrite rule taking $t \in \mathbb{N}$ to a number outside $i(x) \cup o(x)$

$$w_r(x) = \lambda t.\ t + 1 + \max\big(i(x) \cup o(x)\big)$$

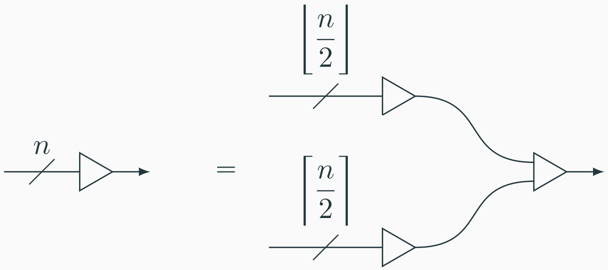
specifies the operator $\mathbf{R}_{\mathbb{L}} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$

$$\mathbf{R}_{\mathbb{L}}(x, y) = x \shortparallel (\lambda(I, O, B).\ ((w_r\, x)^{\star}\, I, (w_r\, x)^{\star}\, O, B))^{\star}\, y$$

Taking it for a spin

Multiway merge

$$f(n) = \begin{cases} \mathbf{I} & \text{if } n = 1 \\ \mathbf{Z}^2\mathbf{R}\big(\mathbf{R}\big(f\lfloor n/2 \rfloor, f\lceil n/2 \rceil\big), \textsc{merge}\big) & \text{otherwise} \end{cases}$$



Multiway merge

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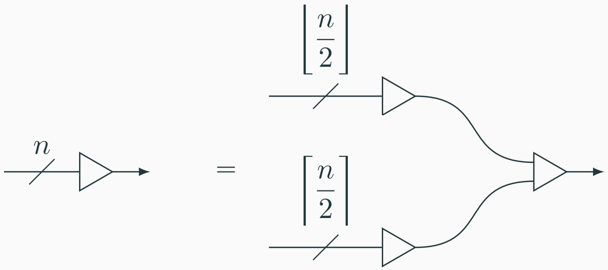
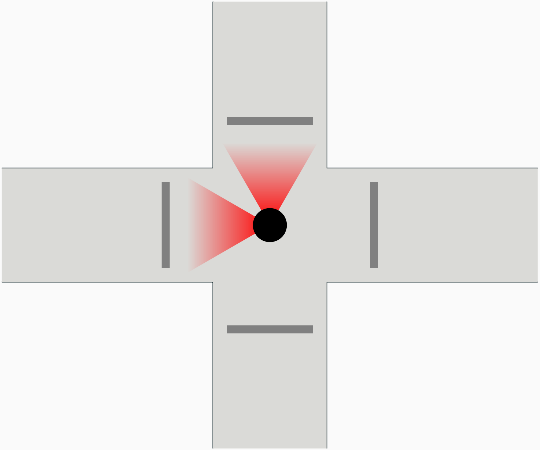


Too easy ?

Traffic control



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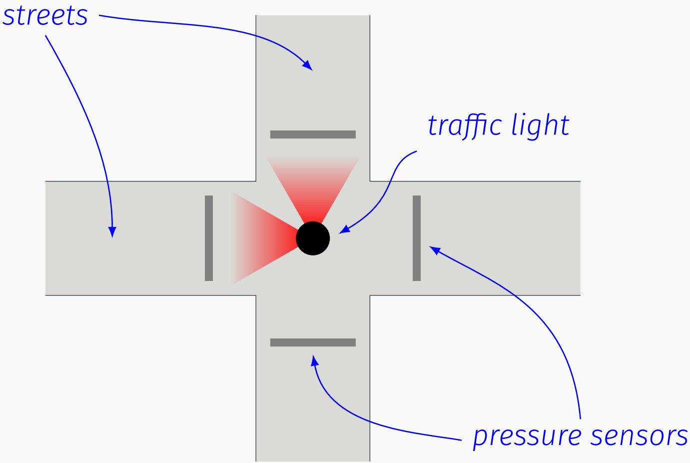

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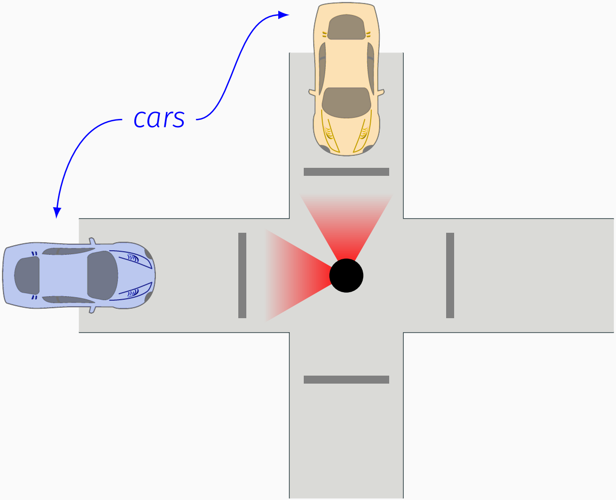

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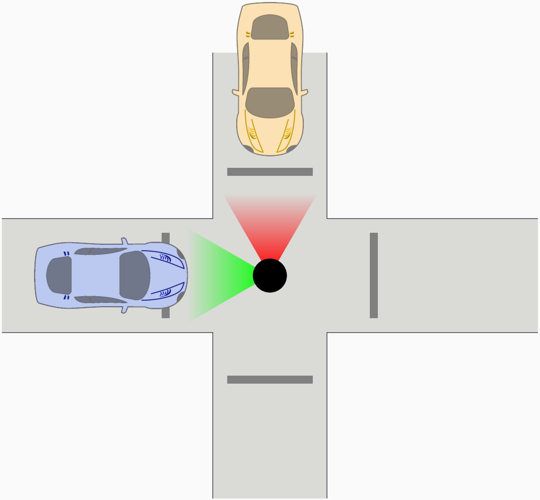

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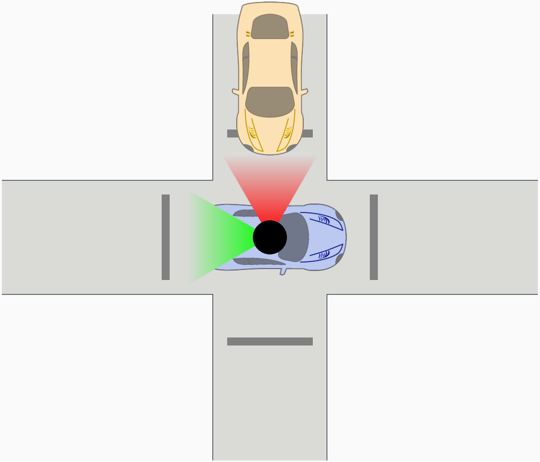

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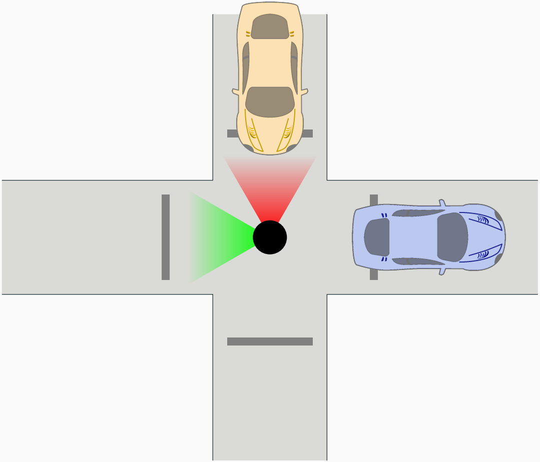

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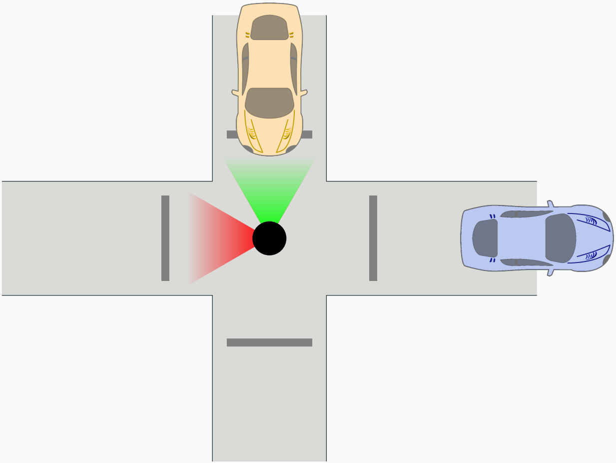

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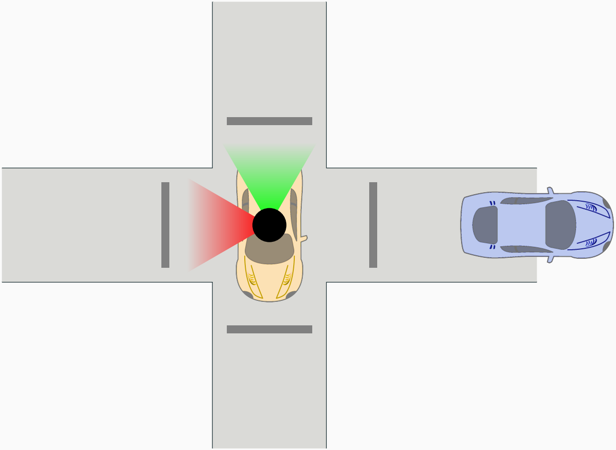

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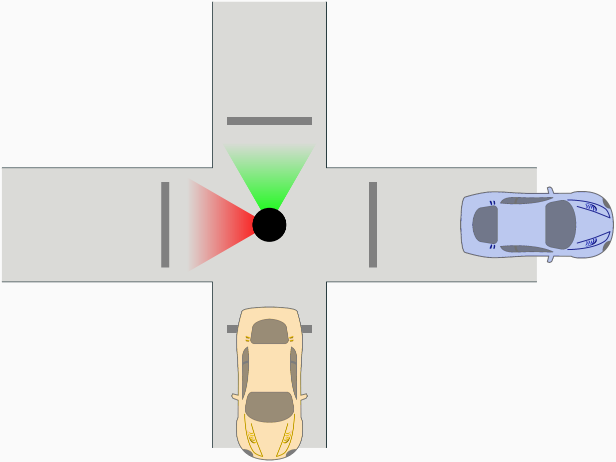

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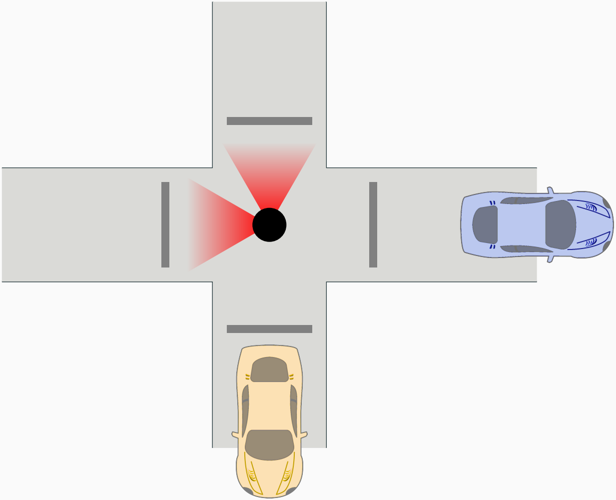
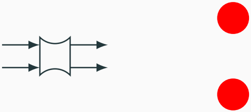


Arbiter circuits

Arbiter circuits



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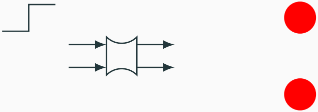

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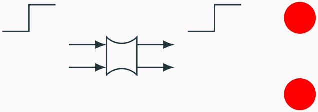

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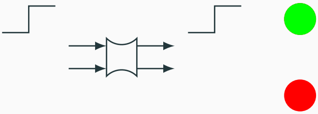

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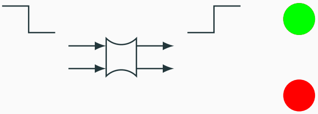

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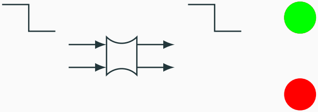

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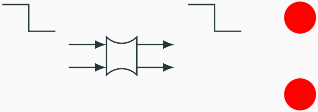

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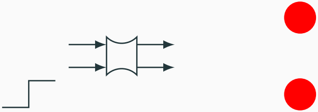

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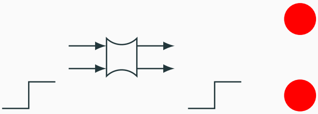

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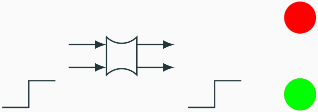

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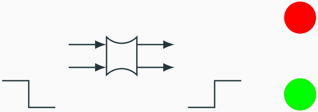

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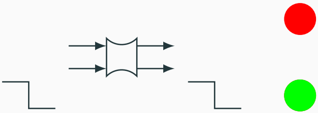

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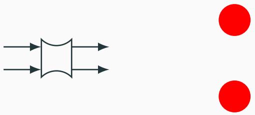

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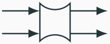

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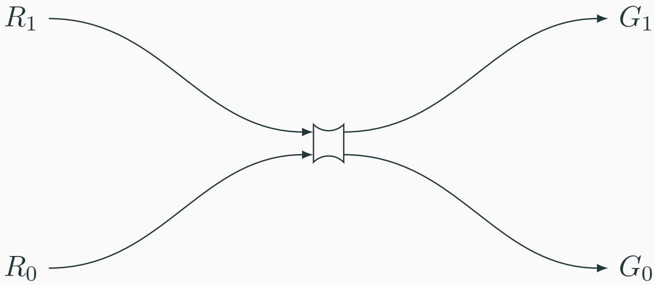

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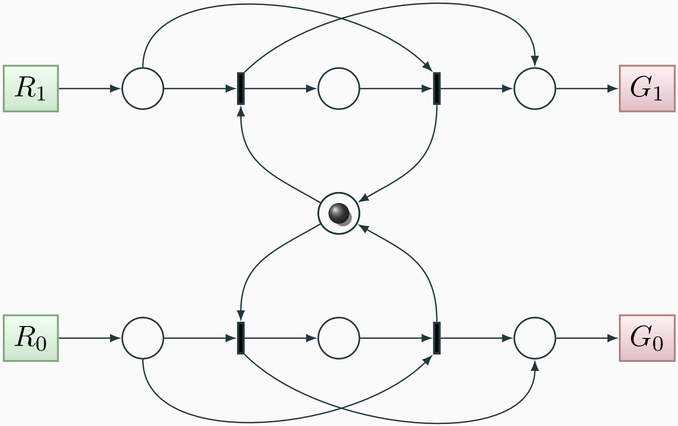

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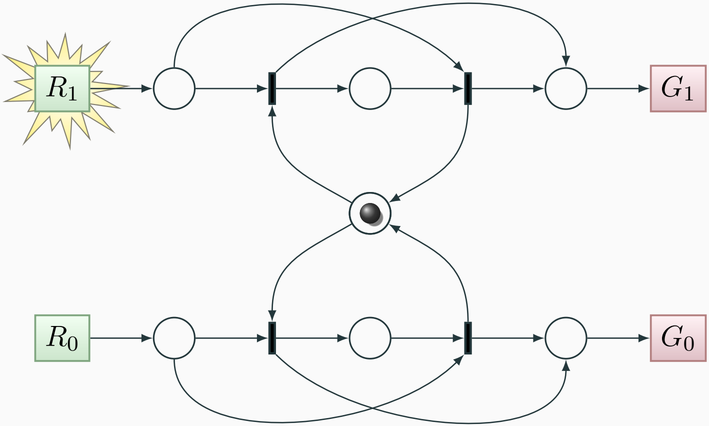

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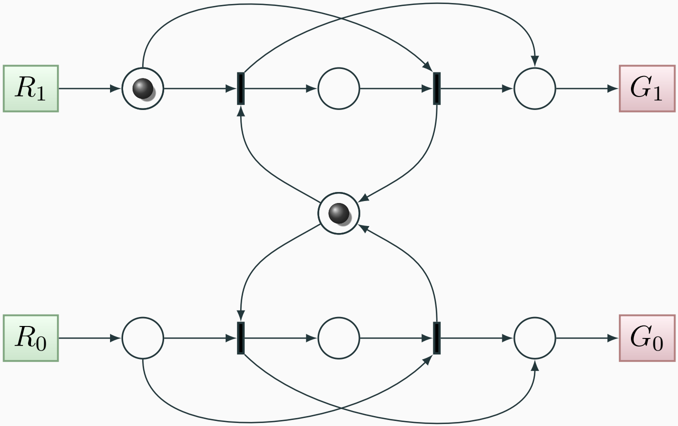

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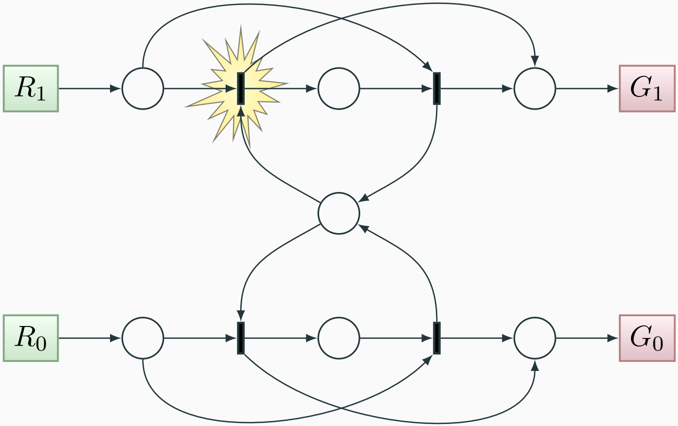

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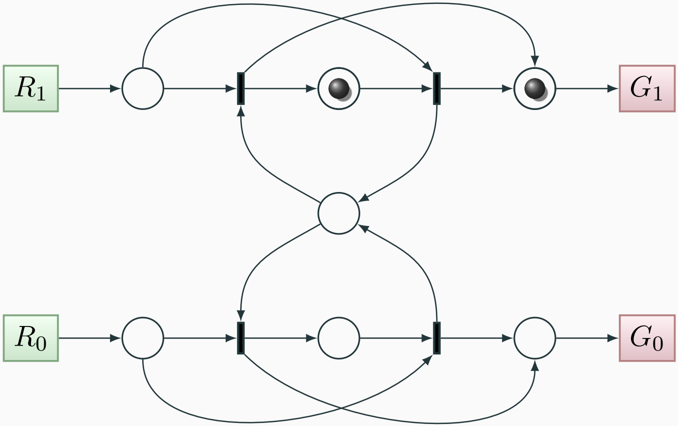

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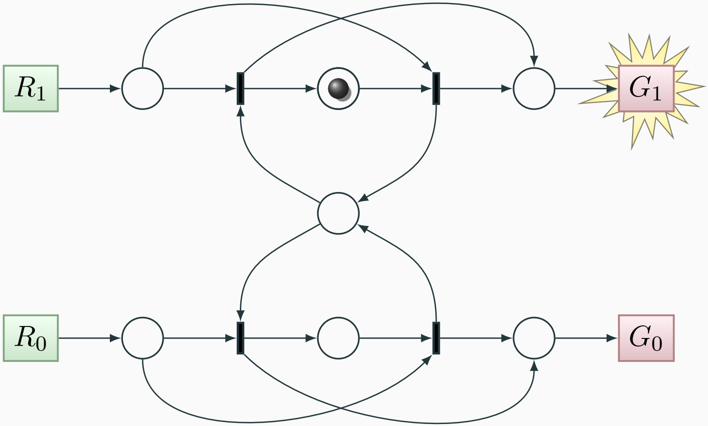

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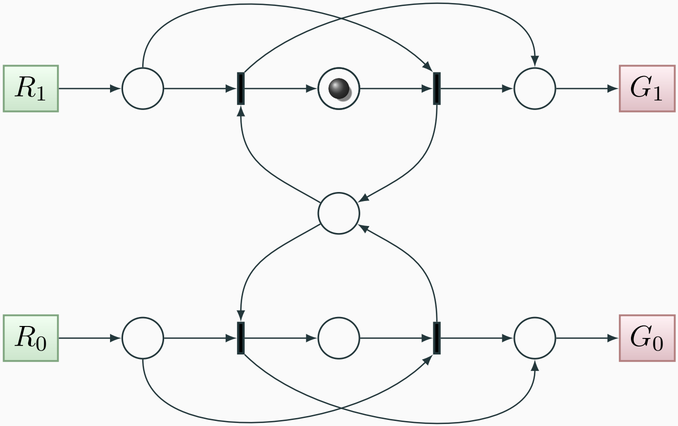

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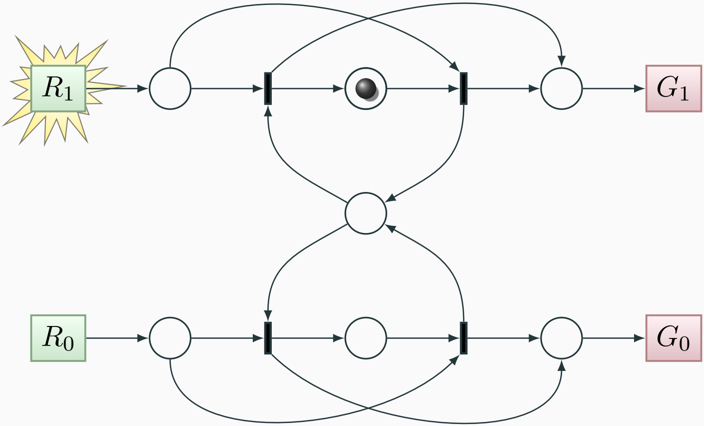
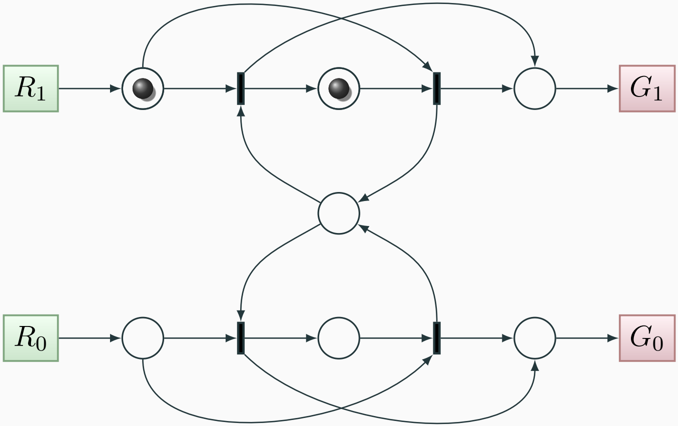
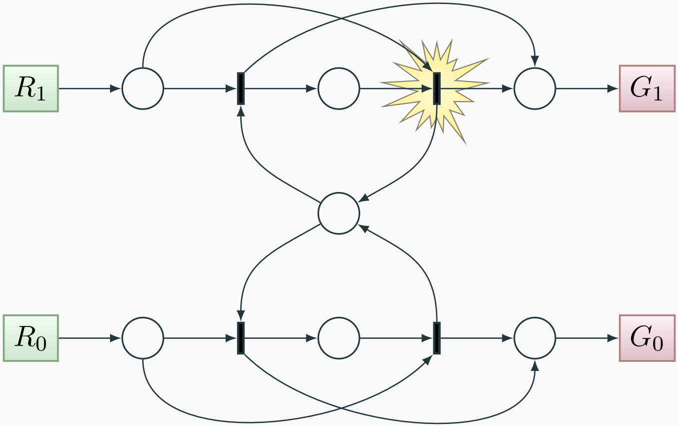



R_1
G_1
R_0
G_0

Arbiter circuits



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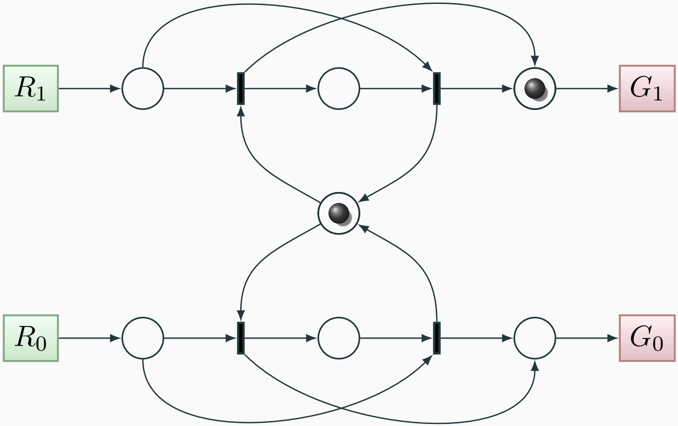

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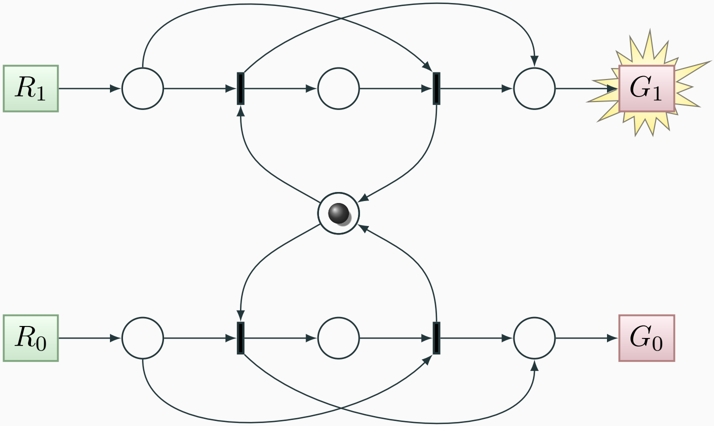
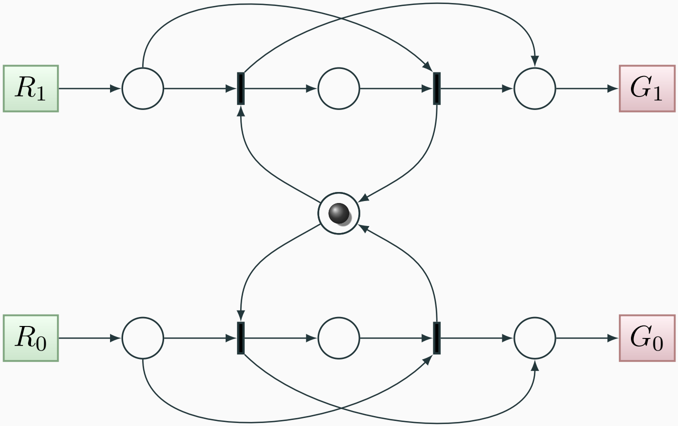
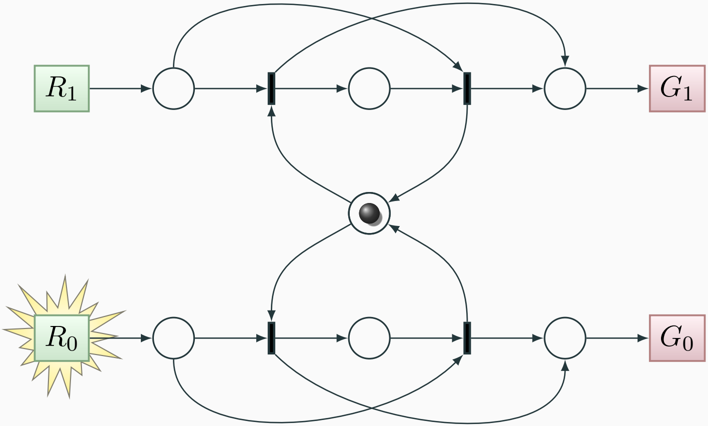


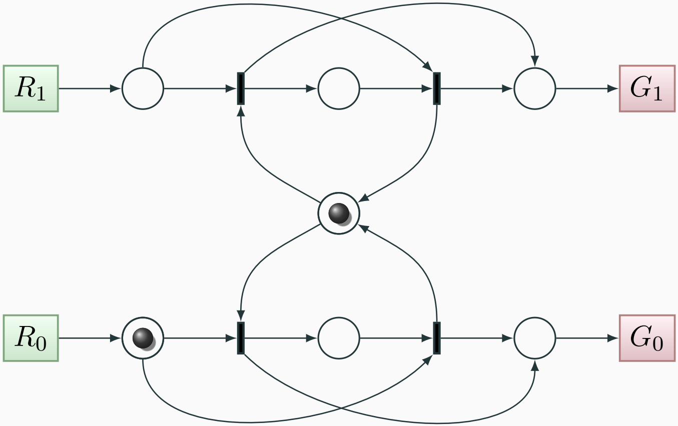
Arbiter circuits



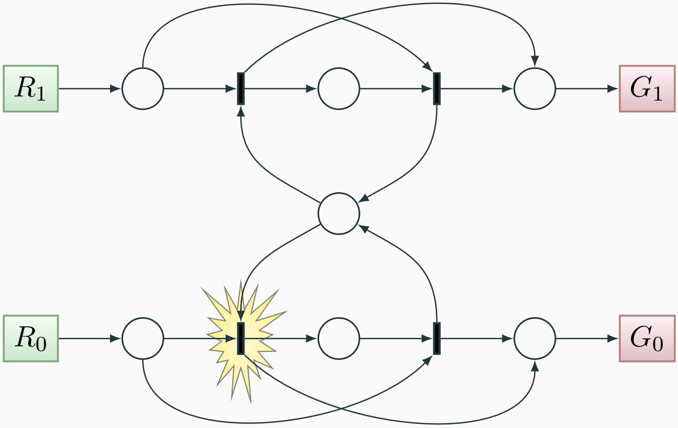
Arbiter circuits



Arbiter circuits



Arbiter circuits



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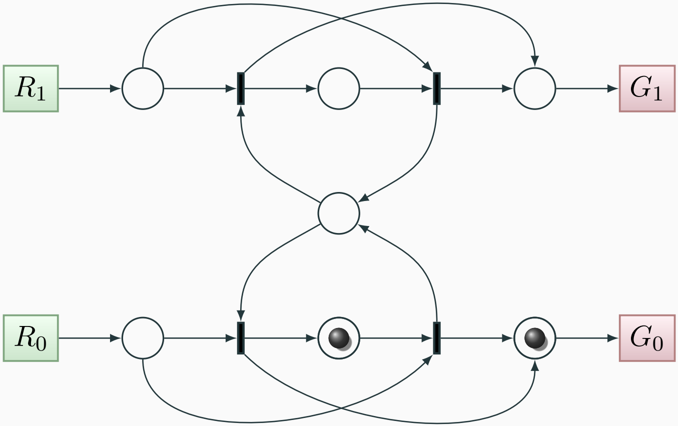

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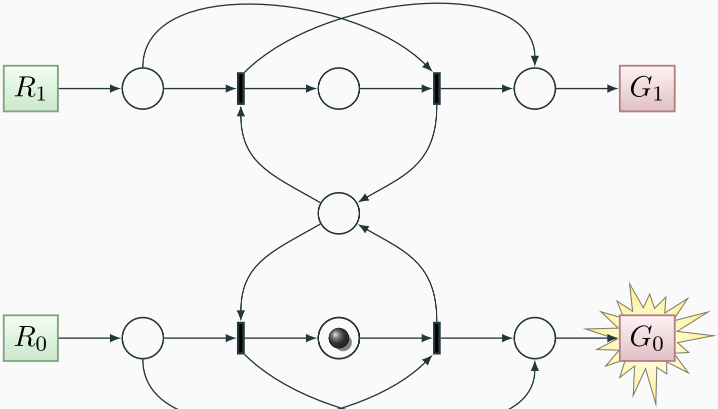
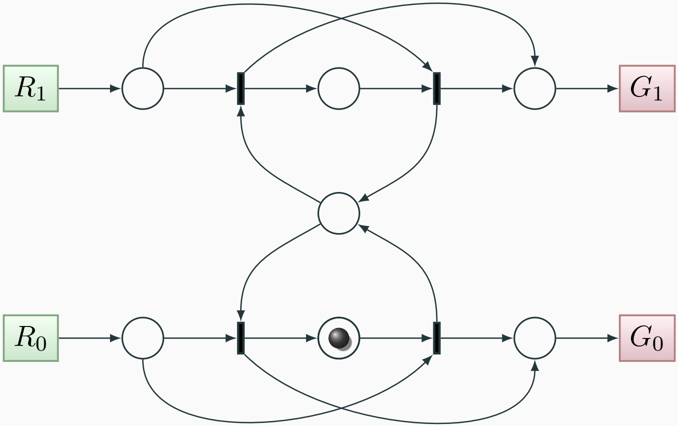
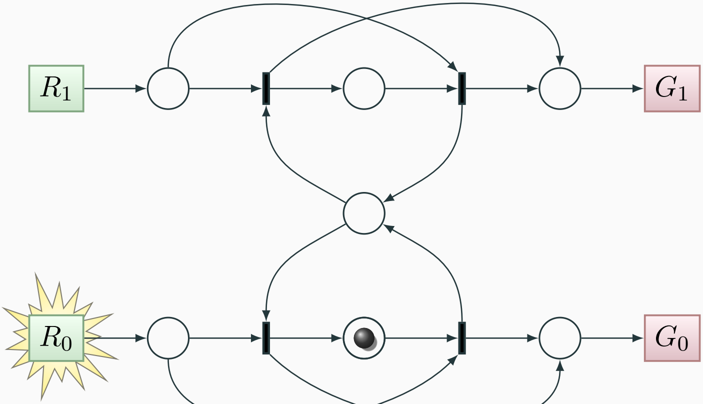


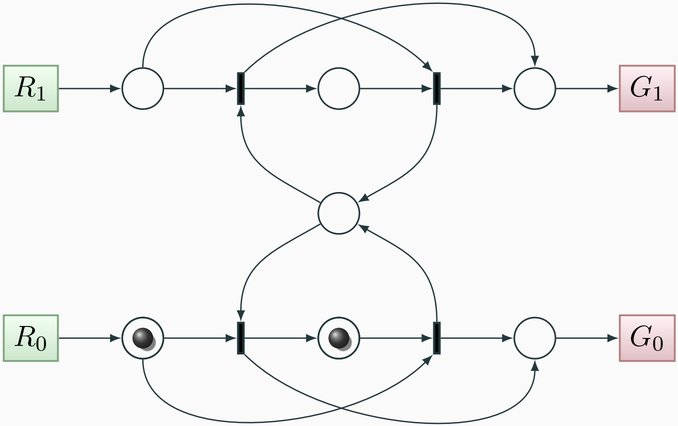
Arbiter circuits



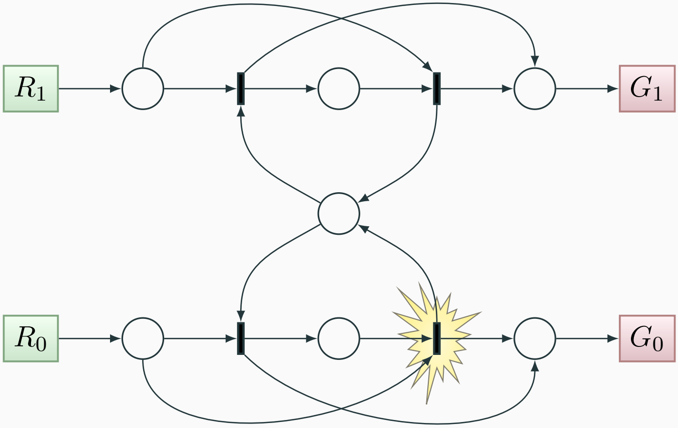
Arbiter circuits



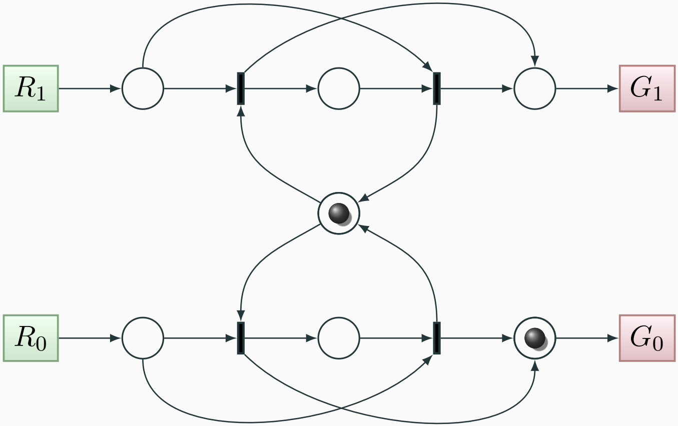
Arbiter circuits



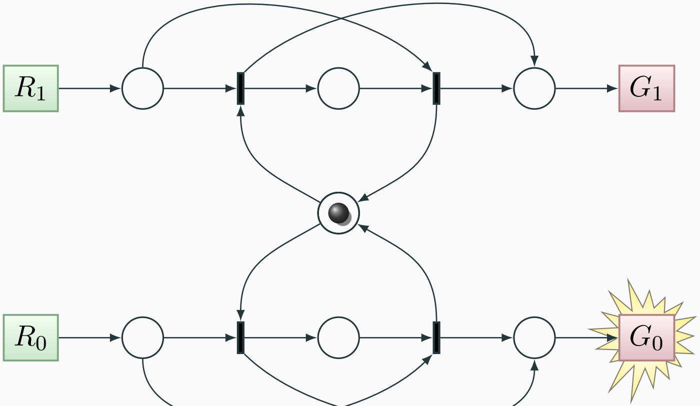
Arbiter circuits



Arbiter circuits



Arbiter circuits



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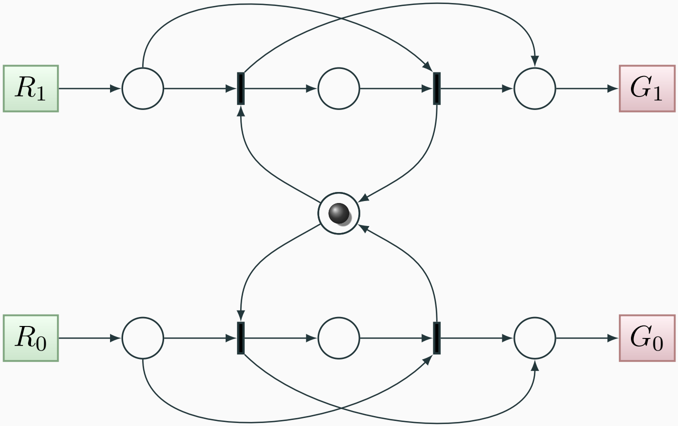
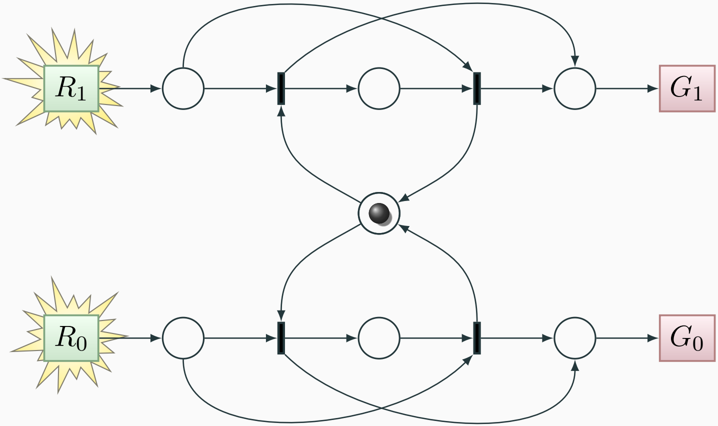
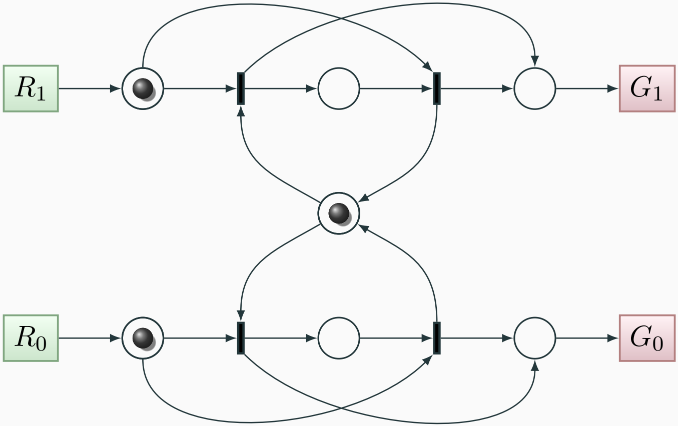


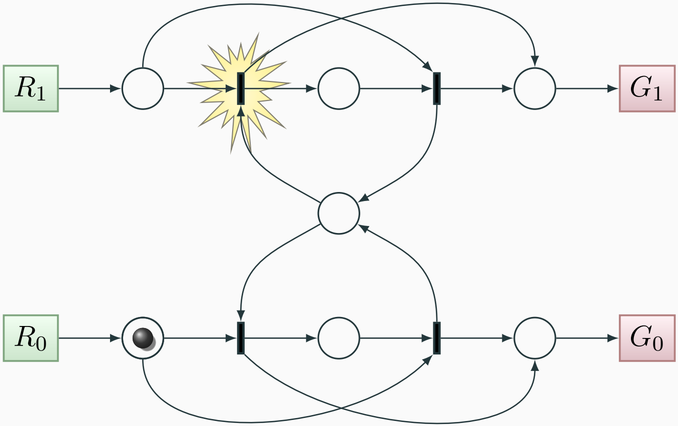
Arbiter circuits



Arbiter circuits



Arbiter circuits



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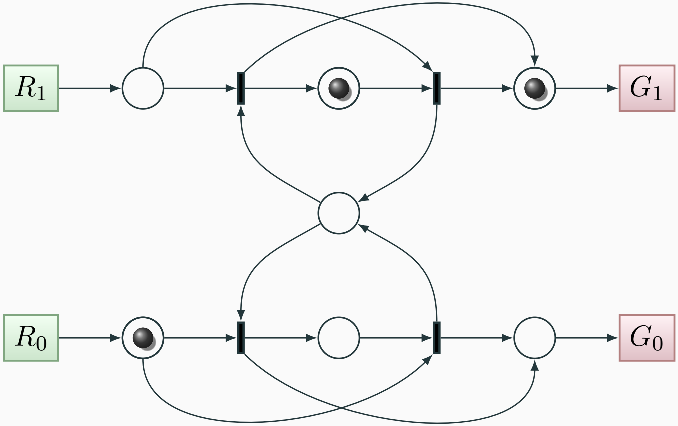

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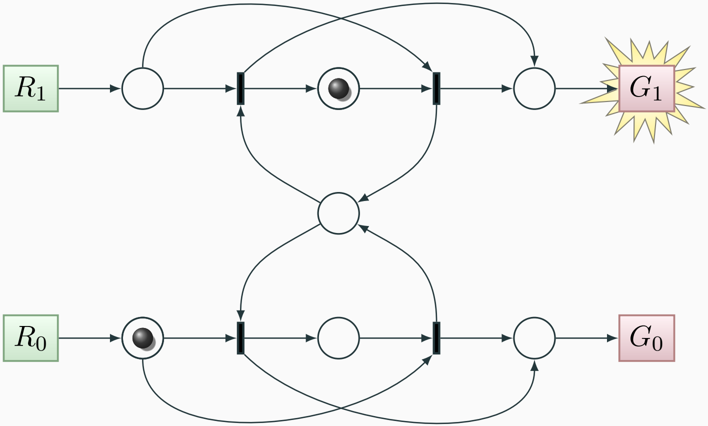

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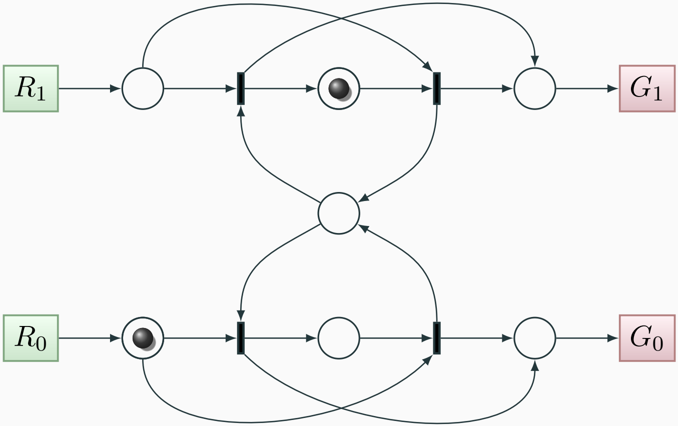
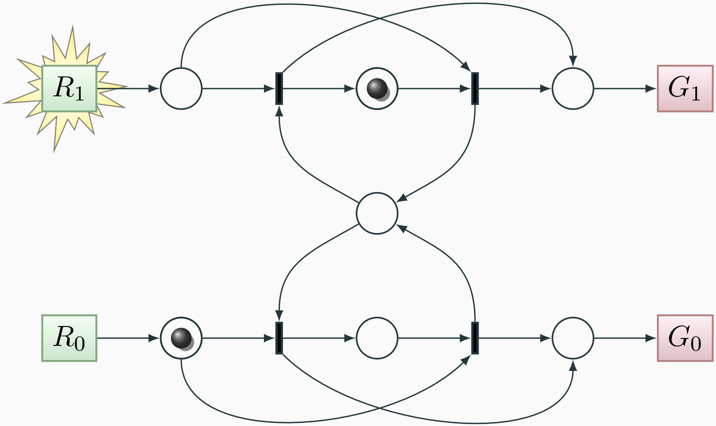
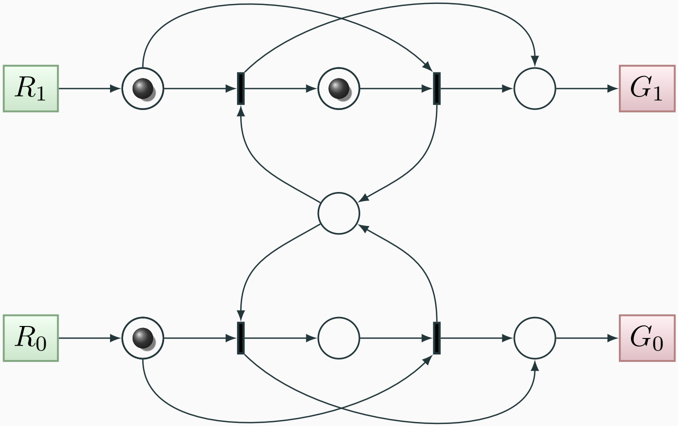


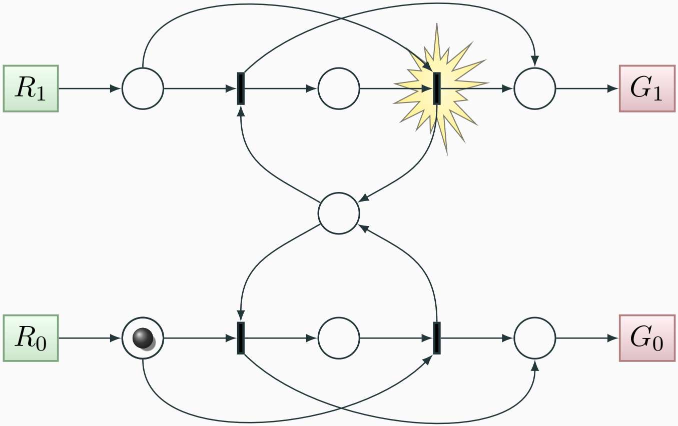
Arbiter circuits



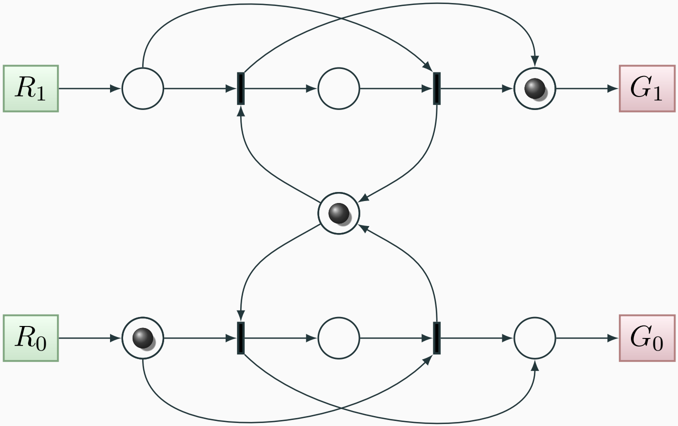
Arbiter circuits



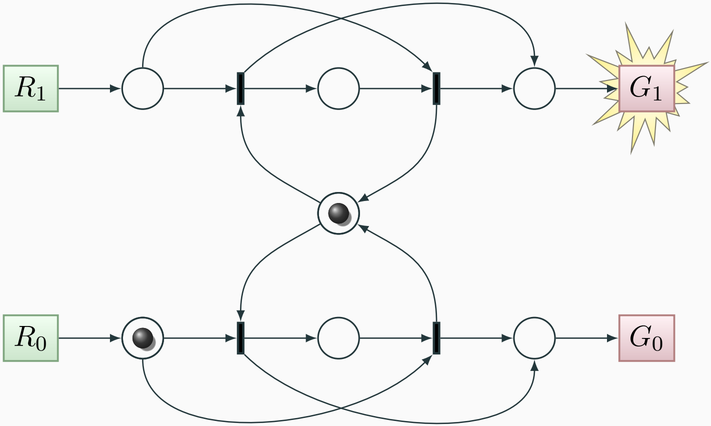
Arbiter circuits



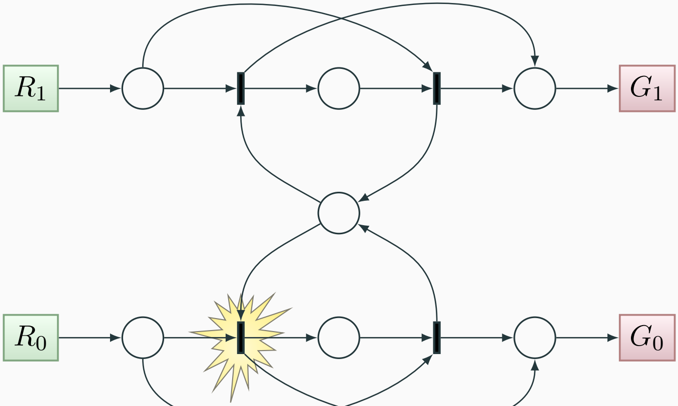
Arbiter circuits



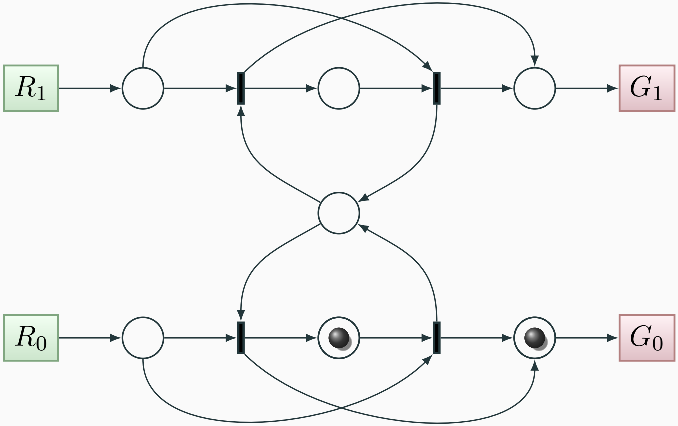
Arbiter circuits



Arbiter circuits



Arbiter circuits



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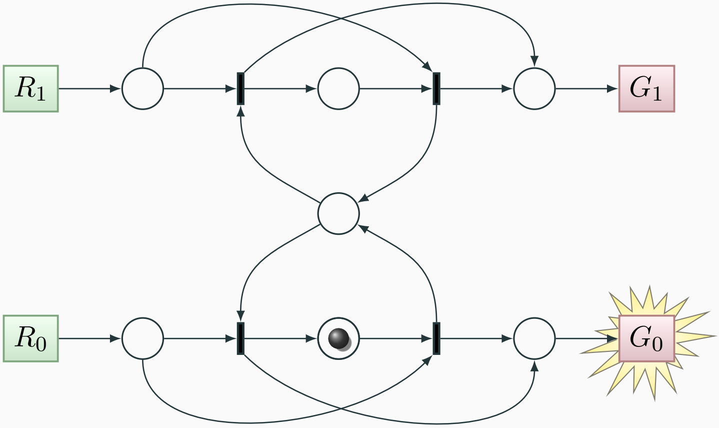

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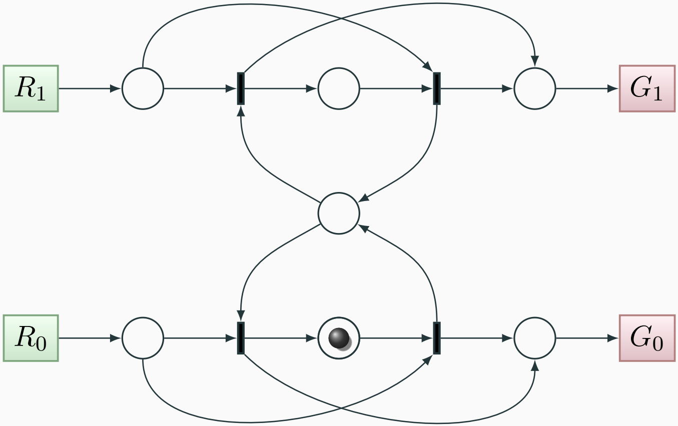

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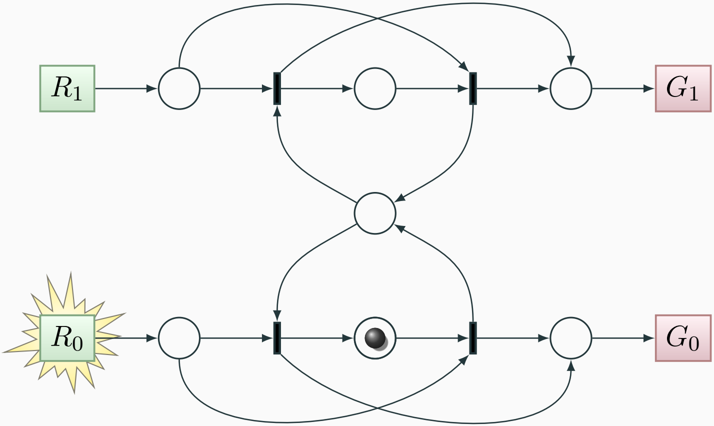

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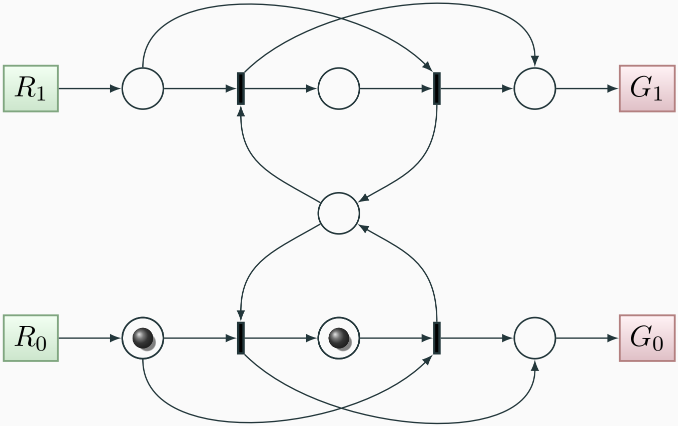

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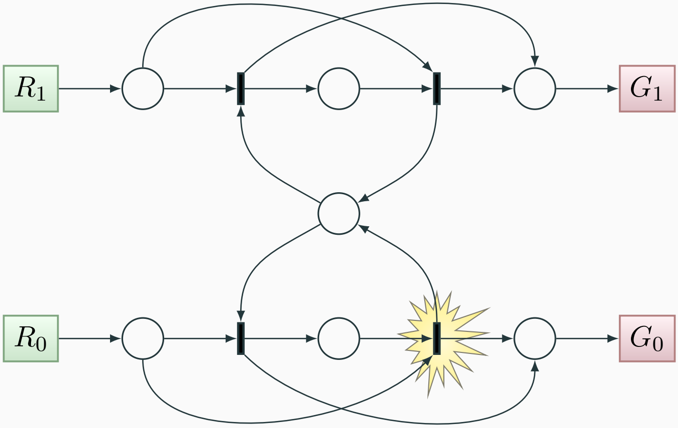

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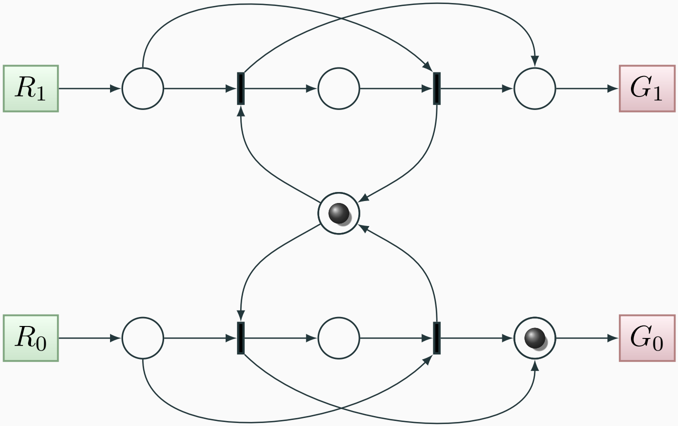
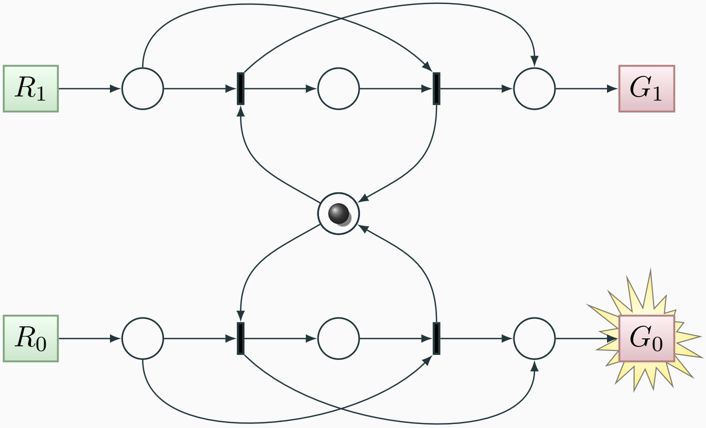


Arbiter circuits



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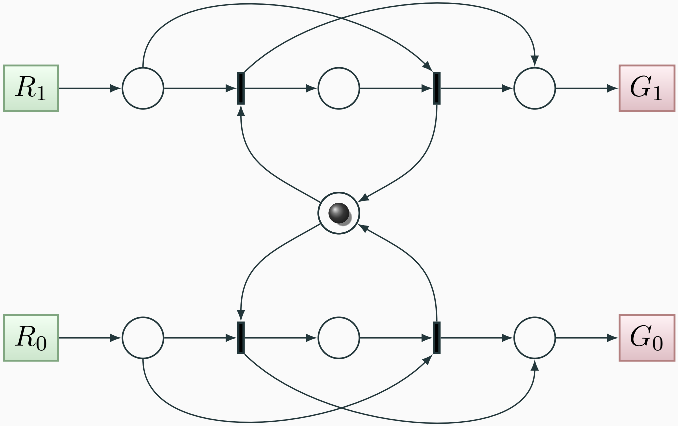

A job for a 4-way arbiter

Each player's button pre-empts the other players.

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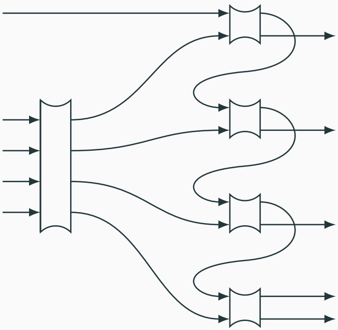
Triangle mesh arbiter



Triangle mesh arbiter

$f(n) \in \mathbb{H}$ expresses an n-way triangle mesh arbiter by

$$\begin{aligned} f(1) &= \mathbf{I} \\ f(n+1) &= \mathbf{S}\mathbf{Z}^n\mathbf{R}\big(f\,n, (\digamma_{\mathbf{I}}\ \lambda(h,t).\,\mathbf{ZR}(\mathbf{ZR}(\textsf{ARB}, \mathbf{S}^{n-h}\,t), \mathbf{I}))\ \iota_n^1\big) \end{aligned}$$

where $\mathbf{S} : \mathbb{H} \to \mathbb{H}$ rolls down the inputs

$$\mathbf{S} = \lambda x.\,\mathbf{ZR}(\mathbf{I}, x)$$

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where $\mathbf{S} : \mathbb{H} \to \mathbb{H}$ rolls down the inputs

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For example, letting $n = 4$ and

$$g = \lambda(h,t).\,\mathbf{ZR}(\mathbf{ZR}(\textsf{ARB}, \mathbf{S}^{4-h}\,t), \mathbf{I})$$

we have

$$f(5) = \mathbf{S}\mathbf{Z}^4\mathbf{R}\big(f\,4, (\digamma_{\mathbf{I}}\,\lambda(h,t).\,\mathbf{ZR}(\mathbf{ZR}(\textsf{ARB}, \mathbf{S}^{4-h}\,t), \mathbf{I}))\,\iota_4^1\big)$$

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$$f(5) = \mathbf{S}\mathbf{Z}^4\mathbf{R}\big(f\,4, g(1, g(2, g(3, g(4, \mathbf{I}))))\big)$$

Triangle mesh arbiter

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we have

$$f(5) = \mathbf{SZZZZR}\big(f\,4, g(1, g(2, g(3, g(4, \mathbf{I}))))\big)$$

Triangle mesh arbiter

evaluating $f(5)$ from the inside out ...

$$\begin{aligned} g(4, \mathbf{I}) &= (\lambda(h,t).\, \mathbf{ZR}(\mathbf{ZR}(\textsf{ARB}, \mathbf{S}^{4-h}\, t), \mathbf{I})\ (4, \mathbf{I}) \\ &= \mathbf{ZR}(\mathbf{ZR}(\textsf{ARB}, \mathbf{S}^{0}\, \mathbf{I}), \mathbf{I}) \\ &\overset{?}{\equiv} \textsf{ARB} \end{aligned}$$

Triangle mesh arbiter

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$$\mathbf{I} = \ t = \ \longrightarrow$$

Triangle mesh arbiter

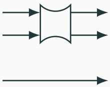
$t =$ ⟶

Triangle mesh arbiter

$$\mathbf{S}^0\,t = \longrightarrow$$

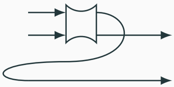
Triangle mesh arbiter

$$\mathbf{R}(\mathsf{ARB}, \mathbf{S}^0\, t) =$$



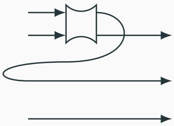
Triangle mesh arbiter

$$\mathbf{ZR}(\mathsf{ARB}, \mathbf{S}^0\, t) =$$



Triangle mesh arbiter

$$\mathbf{R}(\mathbf{ZR}(\textsf{ARB}, \mathbf{S}^0\, t), \mathbf{I}) =$$



Triangle mesh arbiter

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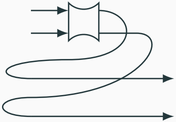
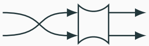


Triangle mesh arbiter

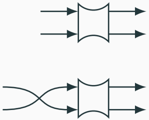
$$g(4, \mathbf{I}) = t =$$ 

Triangle mesh arbiter

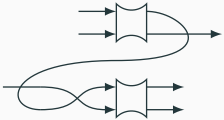
$\mathbf{S}\, t =$ 

Triangle mesh arbiter

$$\mathbf{R}(\mathsf{ARB}, \mathbf{S}\, t) =$$

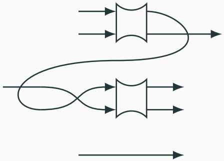


Triangle mesh arbiter

$$\mathbf{ZR}(\mathsf{ARB}, \mathbf{S}\, t) =$$ 

Triangle mesh arbiter

$$\mathbf{R}(\mathbf{ZR}(\textsf{ARB}, \mathbf{S}\,t), \mathbf{I}) =$$



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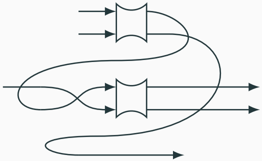
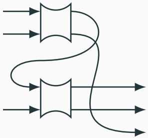
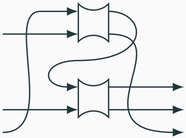


Triangle mesh arbiter

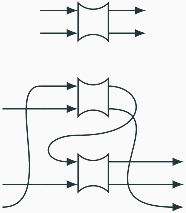
$$g(3, g(4, \mathbf{I})) = t =$$



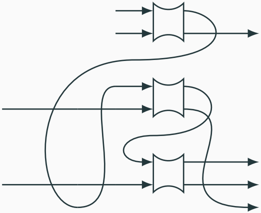
Triangle mesh arbiter

$\mathbf{S}^2\, t =$ 

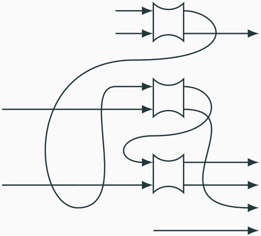
Triangle mesh arbiter

$\mathbf{R}(\mathsf{ARB}, \mathbf{S}^2\, t) =$ 

Triangle mesh arbiter

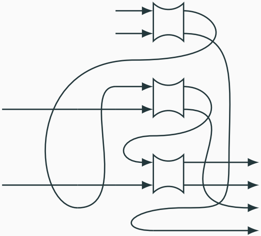
$\mathbf{ZR}(\mathsf{ARB}, \mathbf{S}^2\, t) =$ 

$$\mathbf{R}(\mathbf{ZR}(\mathsf{ARB}, \mathbf{S}^2\, t), \mathbf{I}) =$$



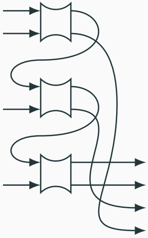
Triangle mesh arbiter

$$\mathbf{ZR}(\mathbf{ZR}(\textsf{ARB}, \mathbf{S}^2\, t), \mathbf{I}) =$$



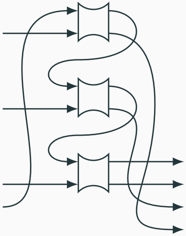
Triangle mesh arbiter

$$g(2, g(3, g(4, \mathbf{I}))) = t =$$



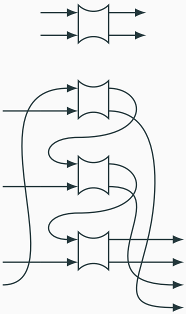
Triangle mesh arbiter

$$\mathbf{S}^3\, t =$$



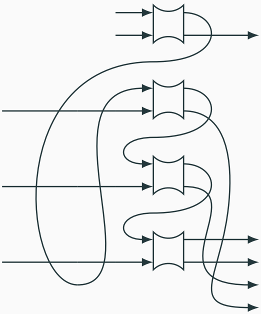
Triangle mesh arbiter

$$\mathbf{R}(\mathsf{ARB}, \mathbf{S}^3\, t) =$$

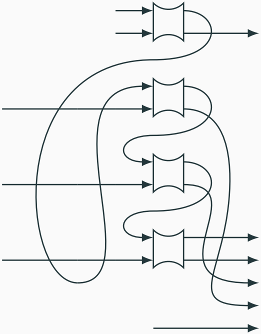


Triangle mesh arbiter

$$\mathbf{ZR}(\mathsf{ARB}, \mathbf{S}^3\, t) =$$

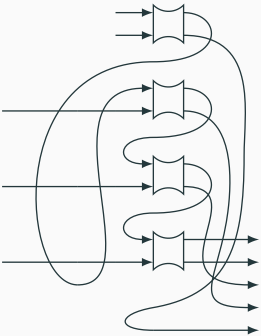


$$\mathbf{R}(\mathbf{ZR}(\mathsf{ARB}, \mathbf{S}^3\, t), \mathbf{I}) =$$



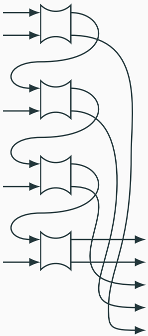
Triangle mesh arbiter

$\mathbf{ZR}(\mathbf{ZR}(\textsf{ARB}, \mathbf{S}^3\, t), \mathbf{I}) =$



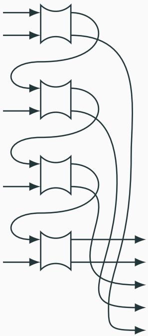
Triangle mesh arbiter

$$g(1, g(2, g(3, g(4, \mathbf{I})))) =$$

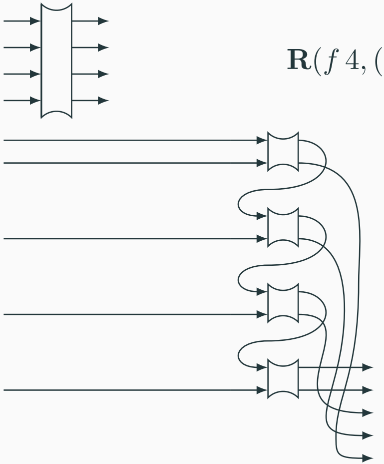


Triangle mesh arbiter

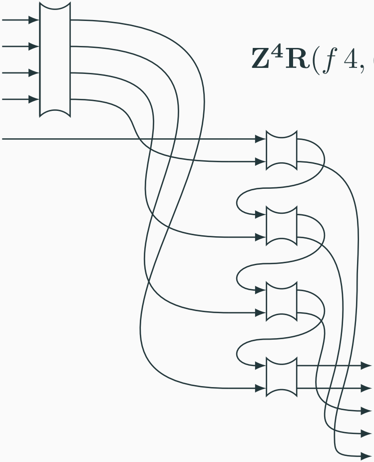
$$(F_{\mathbf{I}}\ g)\ \iota_4^1 =$$



$$\mathbf{R}(f\,4, (F\,\mathbf{I}\,g)\ \iota^1_4)$$

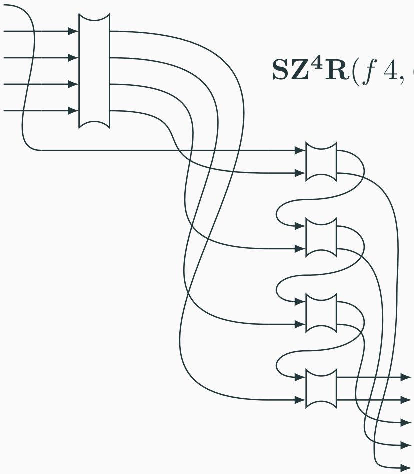


Triangle mesh arbiter



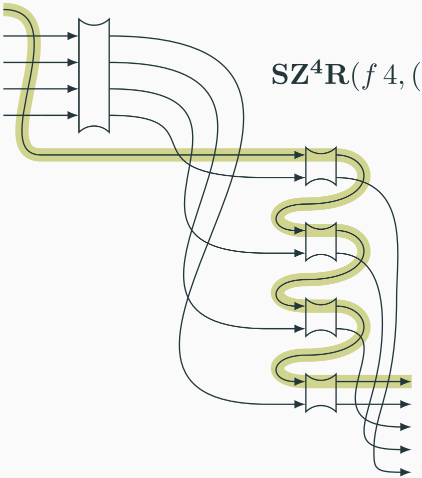
$$\mathbf{Z^4R}(f\,4, (F\;\mathbf{I}\;g)\ \iota_4^1)$$

Triangle mesh arbiter



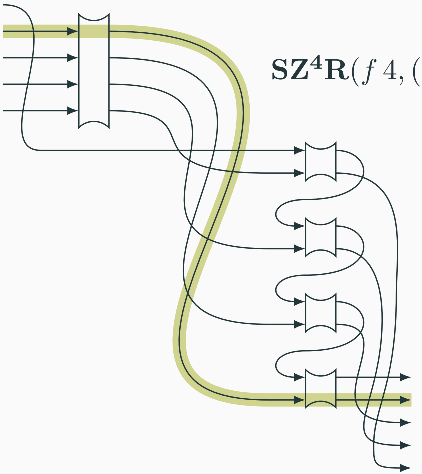
$$\mathbf{SZ^4R}(f\,4, (F\,\mathbf{I}\,g)\ \iota^1_4)$$

Triangle mesh arbiter



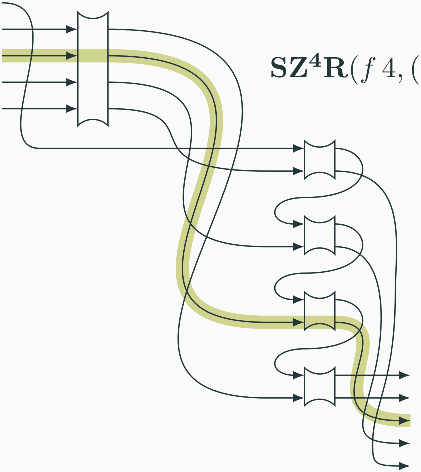
$$\mathbf{S}\mathbf{Z}^{\mathbf{4}}\mathbf{R}(f\,4, (F\ \mathbf{I}\ g)\ \iota_4^1)$$

Triangle mesh arbiter



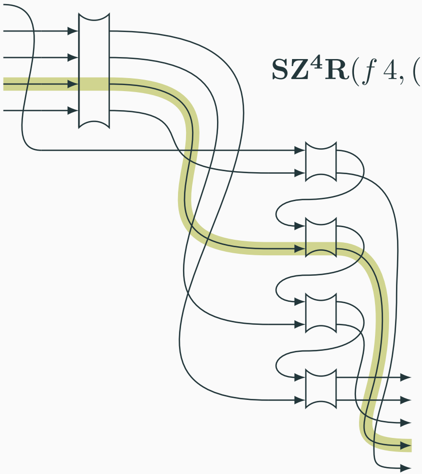
$$\mathbf{SZ^4R}(f\,4, (F\ \mathbf{I}\ g)\ \iota_4^1)$$

Triangle mesh arbiter



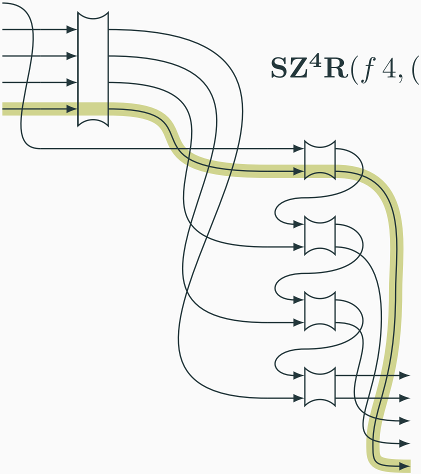
$\mathbf{SZ^4R}(f\,4, (F\,\mathbf{I}\,g)\,\iota_4^1)$

Triangle mesh arbiter



$$\mathbf{SZ^4R}(f\,4, (F\,\mathbf{I}\,g)\ \iota^1_4)$$

Triangle mesh arbiter



$$\mathbf{SZ^4R}(f\,4, (F\,\mathbf{I}\ g)\ \iota_4^1)$$

Summary

- express families of complicated circuits generally
- automatically generate checkable DI semantic models
- automatically generate corresponding netlists

Further reading

- https://www.delayinsensitive.com
  - full details on everything in this presentation
- https://statebox.org
  - overlapping ideas, more ambitious goals
- *Oliver Heaviside : The Life, Work and Times of an Electrical Genius of the Victorian Age*, Paul J. Nahin
  - historical perspective on formal methods